# ...ist. 3  Determinants (p. 201 – 217)

**Defenition:** For $n \geq 2$, the determinant of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of $n$ terms of the form $\pm a_{ij} \det A_{ij}$, with plus and minus signs alternatigng, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of $A$. In symbols:

$$\operatorname{Det}(A) = a_{11}\operatorname{Det}(A_{11}) - a_{12}\operatorname{Det}(A_{12}) + \ldots + (-1)^{n+1} a_{1n}\operatorname{Det}(A_{1n})$$

$$= \sum_{j=1}^{n} (-1)^{1+j} a_{ij} \operatorname{Det}(A_{ij})$$

**Example:** Compute the determinant of:

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Solution: $\operatorname{Det}(A) = 1 \cdot \operatorname{Det}\begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix} - 5 \cdot \operatorname{Det}\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} + 0 \cdot \operatorname{Det}\begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix}$

$$= 1 \cdot (0-2) - 5 \cdot (0-0) + 0 \cdot (-4-0) = -2$$

* Other notation: $\operatorname{Det}(A) = \det\begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix} = \begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix}$

**Theorem 1:** The determinant of an $n \times n$ matrix $A$ can be computed by a cofactor expansion across any row or down any column.

$\Downarrow$

**Theorem 2:** If $A$ is a triangular matrix, then $\det(A)$ is the product of the entries on the main diagonal of $A$.

**Theorem 3:** Row operations:

Let $A$ be a square matrix:

a) If a multiple of one row of $A$ is added to another row to produce a matrix $B$, then $\operatorname{Det}(B) = \operatorname{Det}(A)$

b) If two rows of $A$ are interchanged to produce $B$, then: $\operatorname{Det}(B) = -\operatorname{Det}(A)$

c) If one row of $A$ is multiplied by $k$ to produce $B$, then: $\operatorname{Det}(B) = k \cdot \operatorname{Det}(A)$

Theorem 4: A square matrix is invertible if and only if the determinant of that matrix is <u>not</u> zero

Theorem 5: If $A$ is an $n \times n$ matrix, then: $Det(A) = Det(A^T)$

Theorem 6: Multiplicative property

If $A$ and $B$ are $n \times n$ matrices, then:

$Det(AB) = (Det(A))(Det(B))$

Afsl. 4 Vector Spaces (p. 231 - 293)

Defenition: A vector space is a nonempty set $V$ of objects, called vectors, on which are defined two operations, called addition and multiplication by scalers (real numbers), subject to the ten rules listed below. The rules must hold for all vectors $u, v,$ and $w$ and for all scalars $c$ and $d$.

1) The sum of $u$ and $v$, $u+v$, is in $V$

2) $u+v = v+u$

3) $(u+v)+w = u+(v+w)$

4) There is a zero vector $0$ in $V$ such that $u+0 = u$

5) For each $u$ in $V$, there is a vector $-u$ in $V$ such that $u+(-u) = 0$

6) The scaler multiple of $u$ by $c$, denoted by $cu$, is in $V$.

7) $c(u+v) = cu + cv$

8) $(c+d)u = cu + du$

9) $c(du) = (cd)u$

10 $1u = u$

Defenition: A subspace of a vector space $V$ is a subset $H$ of $V$ that has three properties:

a) The zero vector of $V$ is in $H$

b) $H$ is closed under vector addition. That is, for each $u$ and $v$ in $H$, the sum $u+v$ is in $H$.

c) $H$ is closed under multiplication by scalars. That is, for each $u$ in $H$ and each scalar $c$, the vector $cu$ is in $H$

Theorem 1 If $v_1, \ldots, v_p$ are in a vector space $V$, then $\text{Span}\{v_1, \ldots, v_p\}$ is a subspace of $V$.

**Defenition:** The null space of an $m \times n$ matrix $A$, written as $Nul(A)$, is the set of all solutions to the homogeneous equation $Ax = 0$. In set notation:

$$Nul(A) = \{ x : x \text{ is in } \mathbb{R}^n \text{ and } Ax = 0 \}$$

**Theorem 2:** The null space of an $m \times n$ matrix $A$ is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $Ax = 0$ of $m$ homogeneous linear equations in $n$ unkrows is a subspace of $\mathbb{R}^n$.

**Defenition** The column space of an $m \times n$ matrix $A$, written as $Col(A)$, is the set of all linear combinations of the columns of $A$.
If $A = [a_1 \; \dots \; a_n]$, then

$$Col(A) = Span\{a_1, \dots, a_n\}$$

**Theorem 3:** The column space of an $n \times m$ matrix $A$ is a subspace of $\mathbb{R}^m$.

**Defenition:** A <u>linear transformation</u> $T$ from a vector space $V$ into a vector space $W$ is a rule that assigns to each vector $x$ in $V$ a unique vector $T(x)$ in $W$, such that

i) $T(u+v) = T(u) + T(v)$ for all $u, v$ in $V$

ii) $T(cu) = cT(u)$ for all $u$ in $V$ and all scalars $c$.

**Theorem 4:** An indexed set $\{v_1, \dots, v_p\}$ of two or more vectors, with $v_1 \neq 0$, is linearly dependent if and only if some $v_j$ (with $j > 1$) is a linear combination of the preceding vectors. $v_1, \dots, v_{j-1}$.

**Defenition:** Let $H$ be a subspace of a vector space $V$. An indexed set of vectors $\mathcal{B} = \{b_1, \dots, b_p\}$ in $V$ is a <u>basis</u> for $H$ if

i) $\mathcal{B}$ is a linearly independent set

ii) The subspace spanned by $\mathcal{B}$ coincides with $H$; that is:

$$H = Span\{b_1, \dots, b_p\}$$

Theorem 5: The spanning Set theorem

Let $S = \{v_1, \dots, v_p\}$ be a set in $V$, and let $H = \mathrm{Span}\{v_1, \dots, v_p\}$.

a) If one of the vectors in $S$ - say $v_k$ - is a linear combination of the remaining vectors in $S$, then the set formed from $S$ by removing $v_k$ still spans $H$.

b) If $H \neq 0$, some subset of $S$ is a basis for $H$.

Theorem 6: The pivot columns of a matrix $A$ form a basis for $\mathrm{Col}(A)$.

Theorem 7: The unique Represetntation Theorem

Let $B = \{b_1, \dots, b_n\}$ be a basis for a vector space $V$. Then for each $x$ in $V$, there exists a unique set of scalars $c_1, \dots, c_n$ such that:

$$x = c_1 b_1 + \dots + c_n b_n$$

Defenition: Suppose $B = \{b_1, \dots, b_n\}$ is a basis for $V$ and $x$ is in $V$. The coordinates of $x$ relative to the basis $B$ (or the $B$ coordinates of $x$) are the weights $c_1, \dots, c_n$ such that $x = c_1 b_1 + \dots + c_n b_n$.

Theorem 8: Let $B = \{b_1, \dots, b_n\}$ be a basis for a vector space $V$. Then the coordinate mapping $x \mapsto [x]_B$ is a one-to-one linear transformation from $V$ onto $\mathbb{R}^n$.

Theorem 9: If a vector space $V$ has a basis $B = \{b_1, \dots, b_n\}$ then any set is $V$ containing more than $n$ vectors must be linearly dependent.

Theorem 10: If a vector space $V$ has a basis of $n$ vectors, then every basis of $V$ must consist of exactly $n$ vectors.

**Defenition:** If $V$ is spanned by a finite set, then $V$ is said to be finite-dimensional and the dimension of $V$, written as $Dim(V)$ is the number of vectors in a basis for $V$. The dimension of the zero vector space $\{0\}$ is defined to be zero. If $V$ is not spanned by a finite set, then $V$ is said to be infinite-dimensional.

**Theorem 11:** Let $H$ be a subspace of a finite-dimensional vector space $V$. Any linearly independent set in $H$ can be expanded, if necessary, to a basis for $H$. Also, $H$ finite-dimensional and

$$Dim(H) \leq Dim(V)$$

**Theorem 12:** The Basis Theorem

Let $V$ be a $p$-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly $p$ elements in $V$ is automatically a basis for $V$. Any set of exactly $p$ elements that spans $V$ is automatically a basis for $V$.

**Theorem 13:** If two matrices $A$ and $B$ are row equivalent, then their row spaces are the same. If $B$ is in echelon form, the nonzero rows of $B$ form a basis for the row space of $A$ aswell as for that of $B$.

**Defenition:** The rank of $A$ is the dimension of the column space of $A$.

**Theorem 14:** The rank theorem

The dimensions of the column space and the row space of an $m \times n$ matrix $A$ are equal. This common dimension, the rank of $A$ also equals the number of pivot positions in $A$ and satisfies the equation

$$\text{rank}(A) + \dim(Nul(A)) = n$$

The invertible matrix theorem (continued)

Theorem :

Let $A$ be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that $A$ is an invertible matrix.

m) the columns of $A$ form a basis for $\mathbb{R}^n$

n) $Col(A) = \mathbb{R}^n$

o) $\dim(Col(A)) = n$

p) $rank(A) = n$

q) $Nul(A) = \{0\}$

r) $\dim(Nul(A)) = 0$

Theorem 15: Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ and $\mathcal{C} = \{c_1, \ldots, c_n\}$ be bases of a vector space $V$. Then there is a unique $n \times n$ matrix $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ such that:

$$[x]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} [x]_{\mathcal{B}}$$

The columns of $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ are the $\mathcal{C}$-coördinate vectors of the vectors in basis $\mathcal{B}$. That is:

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = [\,[b_1]_{\mathcal{C}} \; [b_2]_{\mathcal{C}} \; \ldots \; [b_n]_{\mathcal{C}}\,]$$

# HSB. 5 Eigenvalues and Eigenvectors (p. 317 – 369)

**Defenition:** An eigenvector of an $n \times n$ matrix $A$ is a nonzero vector $x$ such that $Ax = \lambda x$ for some scalar $\lambda$. A scalar $\lambda$ is called an eigenvalue of $A$ if there is a nontrivial solution $x$ of $Ax = \lambda x$; such ~~that~~ an $x$ is called an eigenvector corresponding to $\lambda$.

**Theorem 1:** The eigenvalues of a triangular matrix are the entries on its main diagonal

**Theorem 2:** If $v_1, \dots, v_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots, \lambda_r$ of an $n \times n$ matrix $A$, then the set $\{v_1, \dots v_r\}$ is linearly independent.

**Theorem** The invertible matrix theorem (continued)

Let $A$ be an $n \times n$ matrix. Then $A$ is invertible if and only if:

s) The number 0 is <u>not</u> an eigenvalue of $A$

t) The determinant of $A$ is <u>not</u> zero.

**Theorem 3:** Let $A$ and $B$ be $n \times n$ matrices:

a) $A$ is invertible if and only if $\det(A) \neq 0$

b) $\det(AB) = \det(A)\det(B)$

c) $\det(A^T) = \det(A)$

d) If $A$ is triangular, then $\det(A)$ is the product of the entries on the main diagonal of $A$.

e) A row replacement operation on $A$ does not change the determinant. A row interchange changes the sign of the determinant. A row scaling also scales the determinant by the same vector.

**Theorem 4** If $n \times n$ matrices $A$ and $B$ are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities)

**Theorem 5:** The Diagonalization Theorem

An $n \times n$ matrix $A$ is diagonalizable if and only if $A$ has $n$ linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with $D$ a diagonal matrix, if and only if the columns of $P$ are linearly independent eigenvectors of $A$. In this case, the diagonal entries of $D$ are eigenvalues of $A$ that correspond, respectively, to the eigenvectors in $P$.

**Theorem 6:** An $n \times n$ matrix with $n$ distinct eigenvalues is diagonalizable.

**Theorem 7:** Let $A$ be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \dots, \lambda_p$

a/ For $1 \le k \le p$, the dimension of the eigenspace for $\lambda_k$ is less than or equal to the multiplicity of the eigenvalue $\lambda_k$

b/ The matrix $A$ is diagonalizable if and only if the sum of the dimensions of the distinct eigenspaces equals $n$, and this happens if and only if the dimensions of the eigenspace for each $\lambda_k$ equals the multiplicity of $\lambda_k$.

c/ If $A$ is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to $\lambda_k$ for each $k$, then the total collection of vectors in the set $\mathcal{B}_1, \dots, \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.

**Theorem 8:** Diagonal matrix Representation

Suppose $A = PDP^{-1}$ where $D$ is a diagonal $n \times n$ matrix. If $\mathcal{B}$ is the basis for $\mathbb{R}^n$ formed from the columns of $P$, then $D$ is the $\mathcal{B}$-matrix for the transformation $x \mapsto Ax$

HSt 6 Inner product spaces (p. 443 - 451)

Definition. An inner product on a vector space $V$ is a function that, to each pair of vectors $u$ and $v$ in $V$, associates a real number $\langle u, v \rangle$ and satisfies the following axioms, for all $u, v$ and $w$ in $V$ and all scalars $c$:

1/ $\langle u, v \rangle = \langle v, u \rangle$

2/ $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$

3/ $\langle cu, v \rangle = c \langle u, v \rangle$

4/ $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0$ if and only if $u = 0$

A vector space with an inner product is called an inner product space.

Theorem 16: The Cauchy-Schwarz Inequality

For all $u, v$ in $V$,

$$|\langle u, v \rangle| \leq \|u\| \, \|v\|$$

Theorem 17: The Triangle Inequality

For all $u, v$ in $V$,

$$\|u + v\| \leq \|u\| + \|v\|$$