Solutions

Ex. 1 a/ $\underline{p} \in \mathrm{Col}(A) \iff A\underline{x} = \underline{p}$ is consistent

So consider: $\left[\begin{array}{ccc|c} 1 & 2 & a & 3 \\ -2 & 8 & 2 & -6 \\ 2 & -2 & 1 & 7 \end{array}\right] \begin{array}{c} +2 \\ -2 \end{array} \sim$

$$\left[\begin{array}{ccc|c} 1 & 2 & a & 3 \\ 0 & 12 & 2+2a & 0 \\ 0 & -6 & 1-2a & 1 \end{array}\right] +\tfrac{1}{2} \sim \left[\begin{array}{ccc|c} 1 & 2 & a & 3 \\ 0 & 12 & 2+2a & 0 \\ 0 & 0 & 2-a & 1 \end{array}\right]$$

This implies: $\underline{p} \in \mathrm{Col}(A) \iff A\underline{x} = \underline{p}$ is consistent

$\iff 2 - a \neq 0 \iff a \neq 2$

b/ $$A\underline{\varphi} = \begin{bmatrix} 1 & 2 & a \\ -2 & 8 & 2 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 4-2a \\ 0 \\ 0 \end{bmatrix}$$

Using this information: $\underline{\varphi} \in \mathrm{Nul}(A)$

$\iff \underline{\varphi}$ is a solution of $A\underline{x} = \underline{0}$

$\iff 4 - 2a = 0 \iff a = 2$

c/ $$B = \begin{bmatrix} 1 & 2 & b \\ -2 & 4b & 2 \\ b & -2 & 1 \end{bmatrix} \begin{array}{c} +2 \\ -b \end{array} \sim \begin{bmatrix} 1 & 2 & b \\ 0 & 4b+4 & 2+2b \\ 0 & -2-2b & 1-b^2 \end{bmatrix} +\tfrac{1}{2}$$

$$\sim \begin{bmatrix} 1 & 2 & b \\ 0 & 4(b+1) & 2(b+1) \\ 0 & 0 & -b^2+b+2 \end{bmatrix}$$

$$-b^2+b+2 = -(b^2-b-2) = -(b-2)(b+1)$$

Now we establish:

if $b \neq 2$ and $b \neq -1$ then Rank$(B) = 3$

if $b = 2$ then Rank$(B) = 2$

if $b = -1$ then Rank$(B) = 1$

(Recall: Rank(B) = number of pivot positions of B)

Ex. 2 Observe that $H = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ and

since $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is linearly independent

forms $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ a basis for H.

Let $\underline{x} \in H^\perp \iff \underline{x} \perp \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\underline{x} \perp \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\iff \begin{cases} x_1 + x_2 = 0 \\ x_3 = 0 \end{cases} \iff \underline{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

where $x_2 \in \mathbb{R}$

As a consequence:

$H^\perp = \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ and $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for $H^\perp$

Ex. 3 We extend the given two vectors with a third vector $\underline{x}$ such that

$\underline{x} \perp \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ and $\underline{x} \perp \begin{bmatrix} 6 \\ 1 \\ 4 \end{bmatrix}$

(Note: the vectors $\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ 1 \\ 4 \end{bmatrix}$ are orthogonal!)

As a result $\underline{x}$ must satisfy the following Linear system:

$$\begin{cases} x_1 + 2x_2 - 2x_3 = 0 \\ 6x_1 + x_2 + 4x_3 = 0 \end{cases}$$

Consider the corresponding augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 6 & 1 & 4 & 0 \end{array}\right] \Big\downarrow^{-6} \sim \left[\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 0 & -11 & 16 & 0 \end{array}\right]$$

Now we find the solutions $\underline{x} = x_3 \begin{bmatrix} -10 \\ 16 \\ 11 \end{bmatrix}$

where $x_3 \in \mathbb{R}$

and e.g. $\begin{bmatrix} -10 \\ 16 \\ 11 \end{bmatrix}$ is an appropriate third vector in order to obtain an orthogonal basis for $\mathbb{R}^3$ that includes the given two vectors

Ex. 4

Let $\underline{h} \in H = \text{Span}\{\underline{v}, \underline{w}\}$,

then we can write $\underline{h} = c_1 \underline{v} + c_2 \underline{w}$ for certain weights $c_1, c_2 \in \mathbb{R}$.

And $\underline{u} \cdot \underline{h} = \underline{u} \cdot (c_1 \underline{v} + c_2 \underline{w}) = c_1 (\underline{u} \cdot \underline{v}) + c_2 (\underline{u} \cdot \underline{w})$

$= c_1 . 0 + c_2 . 0 = 0$. As a result $\underline{u} \perp \underline{h}$ $\boxtimes$

$\underline{u} \perp \underline{v}$ and $\underline{u} \perp \underline{w}$

Ex. 5 Define $\underline{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\underline{a}_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$ and $\underline{a}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$,

then $\underline{a}_3 = 2\underline{a}_1 + \underline{a}_2$ and $\{\underline{a}_1, \underline{a}_2\}$ is linearly independent. So $\{\underline{a}_1, \underline{a}_2\}$ is a basis for W, but this basis is not orthogonal. So construct an orthogonal basis for W by using the Gram-Schmidt process.

Take $\underline{b}_1 = \underline{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

and $\underline{b}_2 = \underline{a}_2 - \left(\dfrac{\underline{a}_2 \cdot \underline{b}_1}{\underline{b}_1 \cdot \underline{b}_1} \right) \underline{b}_1 = \underline{a}_2 - \left(\dfrac{-2}{2} \right) \underline{b}_1$

$= \underline{a}_2 + \underline{b}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

The component of $\underline{v}$ in W is:

$\underline{w} = \text{proj}_W \underline{v} = \left(\dfrac{\underline{v} \cdot \underline{b}_1}{\underline{b}_1 \cdot \underline{b}_1} \right) \underline{b}_1 + \left(\dfrac{\underline{v} \cdot \underline{b}_2}{\underline{b}_2 \cdot \underline{b}_2} \right) \underline{b}_2$

$= \underline{0} + \dfrac{5}{3} \underline{b}_2 = \begin{bmatrix} 5/3 \\ 5/3 \\ -5/3 \end{bmatrix}$

and the component of $\underline{v}$ in $W^{\perp}$ is

$\underline{v} - \underline{w} = \begin{bmatrix} -2/3 \\ 4/3 \\ 2/3 \end{bmatrix} = \dfrac{2}{3} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

Ex 6 a/ Substituting the data points in the given form yields:

$$\begin{cases} \alpha + \beta = 2 \\ \alpha + \gamma = 3 \\ \alpha + \beta = 3 \\ \alpha + 4\beta - \gamma = 4 \end{cases}$$

or, in matrix form,

$$A \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \underline{b} \quad \text{where } A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 4 & -1 \end{bmatrix}$$

and $\underline{b} = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 4 \end{bmatrix}$

Compute $A^T A = \begin{bmatrix} 4 & 6 & 0 \\ 6 & 18 & -4 \\ 0 & -4 & 2 \end{bmatrix}$

and $A^T \underline{b} = \begin{bmatrix} 12 \\ 21 \\ -1 \end{bmatrix}$ and solve $A^T A \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = A^T \underline{b}$

in order to find the unique least-squares solution $\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1\frac{1}{2} \\ 1 \\ 1\frac{1}{2} \end{bmatrix}$. So the least-squares curve has the equation $y = 1\frac{1}{2} + (t-1)^2 + 1\frac{1}{2} \sin\left(\frac{\pi}{2} t\right)$

b/ The corresponding least-squares error is

$$\left\| \underline{b} - A \begin{bmatrix} 1\frac{1}{2} \\ 1 \\ 1\frac{1}{2} \end{bmatrix} \right\| = \left\| \begin{bmatrix} 2 \\ 3 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 2\frac{1}{2} \\ 3 \\ 2\frac{1}{2} \\ 4 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} \right\|$$

$$= \frac{1}{2}\sqrt{2}$$