Solutions

Ex. 1: An equation of the form $C\underline{x} = \underline{b}$ will have the unique solution $\underline{x} = C^{-1}\underline{b}$ (under the given conditions). Since $NUL(C) = \{\underline{0}\}$ the rank theorem shows that $Rank(C) = 6$ and as a result matrix C is invertible (this also follows from the IMT). So $C\underline{x} = \underline{b} \Leftrightarrow C^{-1}(C\underline{x}) = C^{-1}\underline{b}$
$\Leftrightarrow (C^{-1}C)\,\underline{x} = C^{-1}\underline{b} \Leftrightarrow I_6\,\underline{x} = C^{-1}\underline{b}$
$\Leftrightarrow \underline{x} = C^{-1}\underline{b}$ $\boxtimes$

Ex. 2: Since $NUL(B) = COL(B)$ from the Rank theorem we know $\dim(NUL(B)) = \dim(COL(B)) = 1$.

So $NUL(B) = COL(B) = \text{Span}\left\{\begin{bmatrix} \beta \\ 4\tfrac{1}{2} \end{bmatrix}\right\}$

and $\begin{bmatrix} \beta & -2 \\ 4\tfrac{1}{2} & -\beta \end{bmatrix}\begin{bmatrix} \beta \\ 4\tfrac{1}{2} \end{bmatrix} = \underline{0}$

$$\Leftrightarrow \begin{cases} \beta^2 - 9 = 0 \\ 0 = 0 \end{cases} \Leftrightarrow \beta = \pm 3$$

When $\beta = -3$ indeed $COL(B) = \text{Span}\left\{\begin{bmatrix} -3 \\ 4\tfrac{1}{2} \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix}\right\}$
$= \text{Span}\left\{\begin{bmatrix} -3 \\ 4\tfrac{1}{2} \end{bmatrix}\right\} = NUL(B)$

When $\beta = 3$ indeed $COL(B) = \text{Span}\left\{\begin{bmatrix} 3 \\ 4\tfrac{1}{2} \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \end{bmatrix}\right\}$
$= \text{Span}\left\{\begin{bmatrix} 3 \\ 4\tfrac{1}{2} \end{bmatrix}\right\} = NUL(B)$

So: $NUL(B) = COL(B) \Leftrightarrow \beta = \pm 3$

Ex. 3: $\underline{x} \in H \iff \underline{x} \perp \underline{v} \iff \underline{x} \cdot \underline{v} = 0$

$$\iff \begin{cases} x_1 = x_2 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \\ x_4 \text{ is free} \end{cases} \iff \underline{x} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

where $x_2, x_3, x_4 \in \mathbb{R}$

So $H = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ and since

$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is linearly independent

$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for H.

Ex. 4 a/ Find a basis for ROW(A) first.
After Row reducing A into echelon form we establisch that the Rows

$\underline{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$, $\underline{a}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ -2 \end{bmatrix}$ and $\underline{a}_3 = \begin{bmatrix} 2 \\ -1 \\ 3 \\ -2 \end{bmatrix}$ are

Linearly independent and the bottom Row $\underline{a}_4$ can be expressed as a linear combination of $\underline{a}_1, \underline{a}_2$ and $\underline{a}_3$ $\left(\underline{a}_4 = \underline{a}_1 - \frac{1}{2}\underline{a}_2 + \frac{1}{2}\underline{a}_3\right)$.

So $\{\underline{a}_1, \underline{a}_2, \underline{a}_3\}$ is a basis for ROW(A) and starting with this basis for ROW(A) we construct an orthogonal basis $\{\underline{b}_1, \underline{b}_2, \underline{b}_3\}$ for ROW(A)

Take $\underline{b}_1 = \underline{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

$$\underline{n}_2 = \underline{a}_2 - \left(\frac{\underline{a}_2 \cdot \underline{b}_1}{\underline{b}_1 \cdot \underline{b}_1}\right)\underline{b}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ -2 \end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \\ -1 \end{bmatrix}$$

$$\underline{b}_2 = -\underline{n}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\underline{n}_3 = \underline{a}_3 - \left(\frac{\underline{a}_3 \cdot \underline{b}_1}{\underline{b}_1 \cdot \underline{b}_1}\right)\underline{b}_1 - \left(\frac{\underline{a}_3 \cdot \underline{b}_2}{\underline{b}_2 \cdot \underline{b}_2}\right)\underline{b}_2$$

$$= \begin{bmatrix} 2 \\ -1 \\ 3 \\ -2 \end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 2 \\ -2 \end{bmatrix}$$

$$\underline{b}_3 = -\frac{1}{2}\underline{n}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

Component $\underline{v} \in \mathrm{ROW}(A)$ is the orthogonal projection of $\underline{d}$ onto $\mathrm{ROW}(A)$ so

$$\underline{v} = \left(\frac{\underline{d} \cdot \underline{b}_1}{\underline{b}_1 \cdot \underline{b}_1}\right)\underline{b}_1 + \left(\frac{\underline{d} \cdot \underline{b}_2}{\underline{b}_2 \cdot \underline{b}_2}\right)\underline{b}_2 + \left(\frac{\underline{d} \cdot \underline{b}_3}{\underline{b}_3 \cdot \underline{b}_3}\right)\underline{b}_3$$

$$= \frac{1}{3}\underline{b}_1 + \frac{14}{3}\underline{b}_2 + \frac{5}{3}\underline{b}_3 = \frac{1}{3}\begin{bmatrix} 1+14 \\ 1+5 \\ 14-5 \\ -1+14+5 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix}$$

and $\underline{w} = \mathrm{proj}_{\mathrm{ROW}(A)^{\perp}}(\underline{d}) = \underline{d} - \underline{v} = \begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$

b/ $\dim(\mathrm{NUL}(A^T)) = \dim(\mathrm{COL}(A)^{\perp})$

$$= 4 - \dim(\mathrm{COL}(A)) \underbrace{=}_{\boxed{\text{A has 3 pivot positions}}} 4 - 3 = 1$$

Ex. 5: Substituting the data points in the given form yields:

$$\begin{cases} \alpha - \beta = 0 \\ 2\alpha + \beta = 2 \\ 3\alpha - \beta = 4 \\ 4\alpha + \beta = 5 \end{cases}$$ OR, in matrix form,

$$A\begin{bmatrix}\alpha \\ \beta\end{bmatrix} = \underline{b} \quad \text{where } A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 3 & -1 \\ 4 & 1 \end{bmatrix} \text{ and } \underline{b} = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 5 \end{bmatrix}.$$

Compute $A^T A = \begin{bmatrix} 30 & 2 \\ 2 & 4 \end{bmatrix}$, $A^T \underline{b} = \begin{bmatrix} 36 \\ 3 \end{bmatrix}$

and solve $A^T A \begin{bmatrix}\alpha \\ \beta\end{bmatrix} = A^T \underline{b}$ to find the unique least-squares solution $\begin{bmatrix}\alpha \\ \beta\end{bmatrix} = \begin{bmatrix} 69/58 \\ 9/58 \end{bmatrix}$

So the least-squares curve has the equation

$$y = \frac{69}{58}\sqrt{x} + \frac{9}{58}\cos(\pi x)$$