Solutions

Ex 1 Consider the augmented matrix of the system.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & f \\ 1 & a+1 & 3 & g \\ 0 & b & c & h \end{array}\right] \begin{array}{c} -1 \\ \downarrow \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & f \\ 0 & a & 2 & g-f \\ 0 & b & c & h \end{array}\right] \begin{array}{c} \\ -\frac{b}{a} \\ \downarrow \end{array} \sim$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & f \\ 0 & a & 2 & g-f \\ 0 & 0 & c-2\frac{b}{a} & h-\frac{b}{a}(g-f) \end{array}\right]$$

Now we can draw the conclusion:

The system is consistent for all possible values of f, g and h $\Leftrightarrow$

The matrix of coefficients has a pivot position in each row $\Leftrightarrow$ $c-2\frac{b}{a} \neq 0$

Ex. 2 First we convert the augmented matrix into <u>Reduced</u> echelon form

$$\left[\begin{array}{ccccc|c} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} \uparrow \\ \\ +1 \\ \\ \end{array} \sim$$

$$\left[\begin{array}{ccccc|c} 1 & -3 & 0 & 0 & 9 & 2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} \uparrow \\ +3 \\ \\ \\ \end{array} \sim$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -3 & 5 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow \begin{cases} x_1 = 5 + 3x_5 \\ x_2 = 1 + 4x_5 \\ x_3 \text{ is free} \\ x_4 = 4 - 9x_5 \\ x_5 \text{ is free} \end{cases}$$

A description of the solution set in parametric vector form is:

$$\underline{x} = \begin{bmatrix} 5 \\ 1 \\ 0 \\ 4 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ 4 \\ 0 \\ -9 \\ 1 \end{bmatrix} \quad \text{where } x_3, x_5 \in \mathbb{R}$$

Ex. 3 Given: $5\underline{a}_2 = 2\underline{a}_1 - 4\underline{a}_3$

$$\Leftrightarrow 2\underline{a}_1 - 5\underline{a}_2 - 4\underline{a}_3 = \underline{0}$$

$$\Leftrightarrow \begin{bmatrix} \underline{a}_1 & \underline{a}_2 & \underline{a}_3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -5 \\ -4 \end{bmatrix} = \underline{0}$$

$$\Leftrightarrow A \begin{bmatrix} 2 \\ -5 \\ -4 \end{bmatrix} = \underline{0}$$

So vector $\begin{bmatrix} 2 \\ -5 \\ -4 \end{bmatrix}$ is a solution of $A\underline{x} = \underline{0}$

Ex 4 We solve $c_1 \begin{bmatrix} 1 \\ a \end{bmatrix} + c_2 \begin{bmatrix} a \\ a+2 \end{bmatrix} = \underline{0}$ for c_1 and c_2

So consider:

$$\left[\begin{array}{cc|c} 1 & a & 0 \\ a & a+2 & 0 \end{array}\right] \xrightarrow{-a} \sim \left[\begin{array}{cc|c} 1 & a & 0 \\ 0 & -a^2+a+2 & 0 \end{array}\right]$$

So $\left\{ \begin{bmatrix} 1 \\ a \end{bmatrix}, \begin{bmatrix} a \\ a+2 \end{bmatrix} \right\}$ is linearly independent

$\Longleftrightarrow$ $c_1 \begin{bmatrix} 1 \\ a \end{bmatrix} + c_2 \begin{bmatrix} a \\ a+2 \end{bmatrix} = \underline{0}$ has only the trivial solution for c_1 and c_2

$\Longleftrightarrow -a^2 + a + 2 \neq 0 \Longleftrightarrow -(a^2 - a - 2) \neq 0$

$\Longleftrightarrow -(a-2)(a+1) \neq 0 \Longleftrightarrow a \neq 2$ and $a \neq -1$

Ex. 5 $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ and $T(\underline{x}) = A\underline{x}$ for $\underline{x} \in \mathbb{R}^3$

a. Solve $T(\underline{x}) = \underline{0}$, so solve $A\underline{x} = \underline{0}$

Therefore we consider the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -5 & -7 & 0 \\ -3 & 7 & 5 & 0 \end{array}\right] \overset{+3}{\downarrow} \sim \left[\begin{array}{ccc|c} 1 & -5 & -7 & 0 \\ 0 & -8 & -16 & 0 \end{array}\right] \times\left(-\tfrac{1}{8}\right)$$

$$\sim \left[\begin{array}{ccc|c} 1 & -5 & -7 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \underset{+5}{\uparrow} \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$$\Rightarrow \begin{cases} x_1 = -3x_3 \\ x_2 = -2x_3 \\ x_3 \text{ is free} \end{cases} \Longrightarrow \underline{x} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$$

where $x_3 \in \mathbb{R}$

So $\operatorname{NUL}(T) = \operatorname{Span}\left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} \right\}$

b. Since each <u>row</u> of matrix A has a pivot position transformation T is onto $\mathbb{R}^2$.

c. Since the third <u>column</u> of A is missing a pivot position transformation T is <u>not</u> one-to-one.

Ex. 6

$$\underline{e}_1 \xrightarrow{\text{Reflection}} \underline{e}_1 \xrightarrow{\text{Rotation}} -\underline{e}_1$$

$$\underline{e}_2 \xrightarrow{\text{reflection}} \underline{e}_2 \xrightarrow{\text{Rotation}} \underline{e}_2$$

$$\underline{e}_3 \xrightarrow{\text{reflection}} -\underline{e}_3 \xrightarrow{\text{Rotation}} \underline{e}_3$$

So $T(\underline{e}_1) = -\underline{e}_1$, $T(\underline{e}_2) = \underline{e}_2$, $T(\underline{e}_3) = \underline{e}_3$

and the standard matrix of T is $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Ex. 7

$$A^2 = \begin{bmatrix} a & 0 \\ 1 & d \end{bmatrix}\begin{bmatrix} a & 0 \\ 1 & d \end{bmatrix} = \begin{bmatrix} a^2 & 0 \\ a+d & d^2 \end{bmatrix}$$

$$\text{So } A^2 = A \iff \begin{cases} a^2 = a \iff a(a-1) = 0 \\ a+d = 1 \\ d^2 = d \iff d(d-1) = 0 \end{cases}$$

$$\iff \begin{cases} a=0 \text{ OR } a=1 \\ d=0 \text{ OR } d=1 \\ a+d=1 \end{cases}$$

Now we've the following combinations
$(a=0$ and $d=1)$ OR $(a=1$ and $d=0)$

so $A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ OR $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$.

Ex. d $\begin{bmatrix} 2x & 7 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix} \iff$

$$\begin{bmatrix} 2x & 7 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix}^{-1} = \frac{1}{1}\begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix}$$

$\iff 2x = 4 \iff x = 2$

Ex. g This statement is FALSE, since take for example $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ then

$A^2 = 0$, but $A \neq 0$

Note: All Matrices of the form $A = \begin{bmatrix} a & a \\ -a & -a \end{bmatrix}$

and all matrices of the form

$A = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$, where $a \neq 0$,

provide a counterexample