Solutions

Ex. 1: Consider the augmented matrix
$$\left[\begin{array}{ccc|c} 2 & 4 & 1 & f \\ a & 9 & 3 & g \\ 6 & 12 & b & h \end{array}\right] \begin{array}{c} -\frac{1}{2}a \quad -3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 2 & 4 & 1 & f \\ 0 & 9-2a & 3-\frac{1}{2}a & g-\frac{1}{2}fa \\ 0 & 0 & b-3 & h-3f \end{array}\right]$$

This leads to the conclusion:
The given system is consistent for all possible values of f, g and h $\iff$
Each row of the matrix of co-efficients has a pivot position $\iff$ $a \neq 4\frac{1}{2}$ and $b \neq 3$

Ex. 2: Since $\{\underline{a}_1, \underline{a}_2\}$ is linearly independent and $\{\underline{a}_1, \underline{a}_2, \underline{a}_3\}$ is linearly dependent matrix A has 2 pivot columns (the columns 1 and 2) and the homogeneous linear system $A\underline{x} = \underline{0}$ has 1 free variable (variable x_3). Since $5\underline{a}_1 - 8\underline{a}_2 - \underline{a}_3 = \underline{0}$ (so $A\begin{bmatrix} 5 \\ -8 \\ -1 \end{bmatrix} = \underline{0}$) the solutions of $A\underline{x} = \underline{0}$ are the vectors $\underline{x}_h = c\begin{bmatrix} 5 \\ -8 \\ -1 \end{bmatrix}$ where $c \in \mathbb{R}$.

We also know that $\underline{a}_1 - \underline{a}_2 + 6\underline{a}_3 = \underline{b}$, so $A\begin{bmatrix} 1 \\ -1 \\ 6 \end{bmatrix} = \underline{b}$, and $\underline{x}_p = \begin{bmatrix} 1 \\ -1 \\ 6 \end{bmatrix}$ is a particular solution of $A\underline{x} = \underline{b}$. As a result the general solution of $A\underline{x} = \underline{b}$ is:

$$\underline{x} = \underline{x}_p + \underline{x}_h = \begin{bmatrix} 1 \\ -1 \\ 6 \end{bmatrix} + c\begin{bmatrix} 5 \\ -8 \\ 1 \end{bmatrix} \quad \text{where } c \in \mathbb{R}$$

Ex. 3: True, since $\underline{v}_2 = \frac{1}{2}(\underline{v}_1 + \underline{v}_2) - \frac{1}{2}(\underline{v}_1 - \underline{v}_2)$.

So $\underline{v}_2$ can be written as a linear combination of $\underline{v}_1 + \underline{v}_2$ and $\underline{v}_1 - \underline{v}_2$, and as a result $\underline{v}_2 \in H$. $\boxtimes$

Ex. 4: Establish that: $\begin{bmatrix} 4 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

so $T\left(\begin{bmatrix} 4 \\ 3 \end{bmatrix}\right) \underset{\text{use the linearity of } T}{=} 1\,T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) + 2\,T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

$= \begin{bmatrix} 3 \\ 4 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$

Ex. 5: Transformation T is the matrix transformation associated with standard matrix: $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ -1

$\sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & -1 & -1 \end{bmatrix}$ $+\frac{1}{2}$ $\sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

a) The standard matrix of T doesn't have a pivot position in each row so T is not onto

b) The standard matrix of T doesn't have a pivot position in each column so T is not one-to-one

Ex. 6: $A(B + CX) = D \Leftrightarrow A^{-1}A(B + CX) = A^{-1}D$

$\Leftrightarrow I_n(B + CX) = A^{-1}D \Leftrightarrow B + CX = A^{-1}D$

$\Leftrightarrow -B + B + CX = -B + A^{-1}D \Leftrightarrow$

$CX = A^{-1}D - B \Leftrightarrow C^{-1}(CX) = C^{-1}(A^{-1}D - B)$

$\Leftrightarrow (C^{-1}C)X = C^{-1}A^{-1}D - C^{-1}B$

$\Leftrightarrow I_nX = C^{-1}A^{-1}D - C^{-1}B \Leftrightarrow X = C^{-1}A^{-1}D - C^{-1}B$

Ex. 7: False, under these conditions the rule is

$$(kA)^{-1} = \frac{1}{k} A^{-1}$$

Counterexample: take $k = 2$ and $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

then $(kA)^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$

$$= \frac{1}{k}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{k} A^{-1} \neq kA^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$\boxtimes$

Ex. 8: $(5M)^{-1} = \begin{bmatrix} 5 & 6 \\ 5 & 5 \end{bmatrix} \Leftrightarrow$

$$5M = \begin{bmatrix} 5 & 6 \\ 5 & 5 \end{bmatrix}^{-1} = -\frac{1}{5}\begin{bmatrix} 5 & -6 \\ -5 & 5 \end{bmatrix} \Leftrightarrow$$

$$M = \begin{bmatrix} -1/5 & 6/25 \\ 1/5 & -1/5 \end{bmatrix}$$