Final Exam Linear Algebra TI1206M
24 July 2017, 09.00 – 12.00 h

For the first eight exercises, unless stated otherwise, only the answers are required.
Use the answer form for this. No calculators are allowed. (Thinking may preclude long calculations.)
Credits: **1 – 7**: (18), **8** (8), **9** (10) and **10** (9)

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1. The inverse of the matrix $A = \begin{bmatrix} 3 & 6 & 8 \\ 4 & 2 & 7 \\ 2 & 7 & 6 \end{bmatrix}$ is given by $C = \dfrac{1}{21}\begin{bmatrix} -37 & 20 & 26 \\ -10 & 2 & 11 \\ 24 & -9 & -18 \end{bmatrix}$.

Find the inverses of the matrices $B_1 = \begin{bmatrix} 3 & 4 & 2 \\ 6 & 2 & 7 \\ 8 & 7 & 6 \end{bmatrix}$ and $B_2 = \begin{bmatrix} 6 & 12 & 16 \\ 8 & 4 & 14 \\ 4 & 14 & 12 \end{bmatrix}$.

2. Check whether $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector of the matrix $A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$.
If so, for which eigenvalue?

3. Find the determinant of the matrix $D = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$. (It might be a good idea to do it in two ways and compare the answers.)

4. For the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ it is given that $T\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

and $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$. Find the standard matrix A of this transformation.

5. To find the line $y = \alpha + \beta x$ that best fits (in the least squares sense) the four points (1,0), (3,2), (–2,–1), (–1,–1), which set of (normal) equations must be solved?

6. **a.** Find all (possibly complex) eigenvalues of the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 1 & 0 & 3 \end{bmatrix}$.

b. Is A diagonalizable? Give a short argument.

7. Find the orthogonal projection of the vector $\mathbf{y} = \begin{bmatrix} 9 \\ 2 \\ -4 \end{bmatrix}$ onto the subspace of $\mathbb{R}^3$ spanned

by the set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$. Note that these two vectors are **not** orthogonal.

8. For each statement indicate whether it is true or false and give a short argument (or a counter example) to support your answer.

a. If A is a matrix with the property $A^3 = 0$, then A has the eigenvalue 0.

b. Suppose $\mathbf{v}$ is an eigenvector of both the matrix A and the matrix B.
Then $\mathbf{v}$ will also be an eigenvector of the matrix AB.

c. The product of two 3×3 symmetric matrices is again symmetric.

d. If $A = QR$, where Q is a matrix with orthonormal columns, and R is an invertible upper triangular matrix, then $A(A^TA)^{-1}A^T = QQ^T$.

For the last two exercises you have to give **complete solutions.**
Answer the parts in the correct order!!
Use the first half of a double A4 sheet for exercise 9
and the other half for exercise 10.

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9. Given are a matrix and two vectors:

$$A = \begin{bmatrix} 1 & 1 & -1 & 2 & 5 \\ 1 & 2 & -1 & -1 & -3 \\ 2 & 1 & -2 & 0 & 4 \\ -1 & 1 & 1 & 3 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ -3 \\ 8 \\ -7 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 2 \\ -2 \\ 1 \\ -2 \\ 1 \end{bmatrix}.$$

You may use $\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 2 & 5 & 4 \\ 1 & 2 & -1 & -1 & -3 & -3 \\ 2 & 1 & -2 & 0 & 4 & 8 \\ -1 & 1 & 1 & 3 & 1 & -7 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 1 & -1 & 2 & 5 & 4 \\ 0 & 1 & 0 & -3 & -8 & -7 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$

a. Find all solutions of the equation $A\mathbf{x} = \mathbf{b}$.

b. Check whether $\mathbf{v}$ is in the null space of A.

c. Find a basis for the null space of A.
(Preferably *without* starting a complete row reduction process again.)

d. Find a basis for the column space of A.
Is $\mathbf{b}$ in this column space?

e. Give the definition of an onto function (not necessarily linear) from $\mathbb{R}^n$ to $\mathbb{R}^m$.
(A definition is <u>not</u> an informal description.)

f. Is the linear transformation T with the standard matrix A onto?

10. Given are the matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 2 & 1 \\ 2 & 1 \end{bmatrix}$ and the vector $\mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ -5 \\ -3 \end{bmatrix}$

a. Give an orthogonal basis for the column space of A.

b. Give an orthogonal basis for the orthogonal complement of $\operatorname{Col} A$.

c. Find $\mathbf{v} \in \operatorname{Col} A$ and $\mathbf{w} \in (\operatorname{Col} A)^{\perp}$ such that $\mathbf{b} = \mathbf{v} + \mathbf{w}$.

d. Find the distance of $\mathbf{b}$ to $(\operatorname{Col} A)^{\perp}$.
(Use at least *one sentence* <u>*in words*</u> to explain what is going on.)

SOLUTIONS

1 Note that $B_1 = A^T$ and $B_2 = 2A$ (*Thinking precludes* .... ;-))

$$B_1^{-1} = C^T = \frac{1}{21}\begin{bmatrix} -37 & -10 & 24 \\ 20 & 2 & -9 \\ 26 & 11 & -18 \end{bmatrix} \quad \text{and} \quad B_2^{-1} = \tfrac{1}{2}C = \frac{1}{42}\begin{bmatrix} -37 & 20 & 26 \\ -10 & 2 & 11 \\ 24 & -9 & -18 \end{bmatrix}.$$

2 $A\mathbf{v} = \begin{bmatrix} 10 \\ 5 \end{bmatrix} = 5\mathbf{v}$, so $\mathbf{v}$ is an eigenvector for eigenvalue 5

3 $\det(D) = -2$

4 Obviously, $T(\mathbf{e}_2) = \frac{1}{2}T\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$,

and from $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, it follows that $T(\mathbf{e}_1) = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

Then $\big[T\big] = \big[T(\mathbf{e}_1)\ \ T(\mathbf{e}_2)\big] = \begin{bmatrix} 2 & 1/2 \\ 1 & 1/2 \\ 1 & -1/2 \end{bmatrix}$

5 $\begin{bmatrix} 4 & 1 \\ 1 & 15 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \end{bmatrix}$. Or: $\begin{cases} 4\alpha + \beta = 0 \\ \alpha + 15\beta = 9 \end{cases}$

6a $\lambda_1 = -2$; $\lambda_{2,3} = 2 \pm \sqrt{2}$.

6b Since A has three different real eigenvalues, there will be three independent eigenvectors, so **yes**, A **is** diagonalizable.

7 The solution of $\left[\begin{array}{cc|c} 2 & 1 & 5 \\ 1 & 2 & -2 \end{array}\right]$: $\hat{\mathbf{x}} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$; projection: $A\hat{\mathbf{x}} = \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$.

8a TRUE: $\det(A^3) = (\det A)^3 = 0$, so $\det(A - 0\mathrm{I}) = 0$, from which it follows immediately that A has eigenvalue 0.

Alternative: If $A^3 = 0$ and $A\mathbf{v} = \lambda\mathbf{v}$ for $\mathbf{v} \neq \mathbf{0}$, then on the one hand $A^3\mathbf{v} = 0\mathbf{v} = \mathbf{0}$, and on the other hand $A^3\mathbf{v} = A^2A\mathbf{v} = A^2\lambda\mathbf{v} = \ldots = \lambda^3\mathbf{v}$, so $\lambda^3\mathbf{v} = \mathbf{0}$. Since it was assumed that $\mathbf{v} \neq \mathbf{0}$ it follows that $\lambda^3 = 0$, which implies that $\lambda = 0$.

Quite a few people assumed that $A^3 = 0$ implies $A = 0$, or that A should have 0's on its diagonal. Where did they get that 'wisdom' from??

For instance, $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -3 & 2 \\ -1 & -3 & 2 \end{bmatrix}$
all have the property $A^3 = 0$.

8b TRUE: Simple!! $A\mathbf{v} = \lambda_1\mathbf{v}$ and $B\mathbf{v} = \lambda_2\mathbf{v}$ imply that $AB\mathbf{v} = A\lambda_2\mathbf{v} = \lambda_1\lambda_2\mathbf{v}$, which shows that $\mathbf{v}$ is an eigenvector of AB for the eigenvalue $\lambda_1\lambda_2$.

8c FALSE: e.g. $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} \ldots & 4 & \ldots \\ 2 & \ldots & \ldots \\ \ldots & \ldots & \ldots \end{bmatrix}$.

Almost any example will work, but some people chose complicated examples and then made one or more computation errors :-(
Also 2×2 counterexamples came up too often ....

8d TRUE: use the rules $Q^TQ = I$, and $(AB)^T = B^TA^T$:

$$(QR)\big((QR)^T(QR)\big)^{-1}(QR)^T = (QR)\big(R^T\underline{Q^TQ}R\big)^{-1}(R^TQ^T) = QR\,\big(R^TR\big)^{-1}R^TQ^T =$$

$$= QR\,R^{-1}(R^T)^{-1}R^TQ^T = QQ^T, \quad \text{since} \quad RR^{-1} = I \quad \text{and} \quad (R^T)^{-1}R^T = I.$$

Note that A and Q in general will not be square matrices, so people that talk about A^{-1} or Q^{-1} live in a parallel universe.

9a Further row reducing the already reduced form:

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 2 & 5 & 4 \\ 0 & 1 & 0 & -3 & -8 & -7 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & -1 & 5 & 13 & 11 \\ 0 & 1 & 0 & -3 & -8 & -7 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 3 & 6 \\ 0 & 1 & 0 & 0 & -2 & -4 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Hence x_3 and x_5 can be taken as free variables, and the general solution in vector form becomes

$$\mathbf{x} = \begin{bmatrix} 6 \\ -4 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

9b $A\mathbf{v} = \begin{bmatrix} 0 \\ -4 \\ 4 \\ -8 \end{bmatrix} \neq \mathbf{0}$, so **no**, $\mathbf{v}$ is not in the null space of A.

9c The null space is the homogeneous part of the solution of the system $A\mathbf{x} = \mathbf{b}$.

So no further calculations are needed! A (possible) basis: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

9d We can use the pivot columns of A, which correspond to the pivot columns in the row reduced matrix: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 3 \end{bmatrix} \right\}$

In part **a.** it was shown that $A\mathbf{x} = \mathbf{b}$ is consistent, so **yes**, vector $\mathbf{b}$ is in the column space of A.

9e For any vector $\mathbf{y}$ in $\mathbb{R}^4$ there is at least one vector $\mathbf{x}$ in $\mathbb{R}^5$ for which $T(\mathbf{x}) = \mathbf{y}$.

9f No, it is not: the range of T is the column space of A, and since the latter has dimension three, it is *smaller* than the whole $\mathbb{R}^4$.

10a Gram-Schmidt: $\mathbf{b}_1 = \mathbf{a}_1$,

$$\mathbf{b}_2 = \mathbf{a}_2 - \hat{\mathbf{a}}_1 = \mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{b}_1}{\mathbf{b}_1 \cdot \mathbf{b}_1}\mathbf{b}_1 = \begin{bmatrix} 2 \\ 4 \\ 1 \\ 1 \end{bmatrix} - \frac{10}{10}\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \\ -1 \end{bmatrix}.$$

Answer: $\{\mathbf{b}_1, \mathbf{b}_2\}$.

10b $\text{Dim}(\ (\text{Col}\, A)^{\perp}) = 4 - \text{Dim}\ (\text{Col}\, A) = 2$, so we need: two vectors that are orthogonal to the columns of A, and orthogonal to each other.
One way is: first find a basis for $(\text{Col}\, A)^{\perp}) = \text{Nul}(A^T)$ and then use Gram-Schmidt.
This works a bit awkward here.

Alternative: the vector $\mathbf{a}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$ is certainly orthogonal to $\text{Col}\, A$, and then we just need a vector orthogonal to $\text{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$:

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 0 \\ 2 & 4 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 0 \\ 0 & 2 & -3 & -3 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 4 & 0 \\ 0 & 2 & 0 & -6 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 7 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

We find a second orthogonal vector $\mathbf{a}_4 = \begin{bmatrix} -7 \\ 3 \\ 1 \\ 1 \end{bmatrix}$, and we can take the basis $\{\mathbf{a}_3, \mathbf{a}_4\}$.

10c (e.g.) $\mathbf{v}$ is the projection of $\mathbf{b}$ onto $\text{Col}\, A$, for which we can use the *orthogonal* basis $\{\mathbf{b}_1, \mathbf{b}_2\}$:

$$\mathbf{v} = \frac{\mathbf{b} \cdot \mathbf{b}_1}{\mathbf{b}_1 \cdot \mathbf{b}_1}\mathbf{b}_1 + \frac{\mathbf{b} \cdot \mathbf{b}_2}{\mathbf{b}_2 \cdot \mathbf{b}_2}\mathbf{b}_2 = \frac{-10}{10}\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} + \frac{24}{12}\begin{bmatrix} 1 \\ 3 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -4 \\ -4 \end{bmatrix}, \text{ and then } \mathbf{w} = \mathbf{b} - \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

10d $\mathbf{w}$ is the orthogonal projection of $\mathbf{b}$ onto $(\text{Col}\, A)^{\perp}$.
Thus $\text{dist}(\mathbf{b}, (\text{Col}\, A)^{\perp}) = \text{dist}(\mathbf{b}, \mathbf{w}) = ||\mathbf{b} - \mathbf{w}|| = ||\mathbf{v}|| = \sqrt{1^2 + 5^2 + (-4)^2 + (-4)^2} = \sqrt{58}$.