(Final) Exam Linear Algebra, TI1206M
19 April 2017, 09.00 – 12.00 uur

For most exercises only the answers are required. Use the answer form for exercise 1 to 8. No calculators (nor smart watches nor whatever) are allowed. (Thinking may preclude long calculations.) Credits: exercises 1 – 7: **19** pt, exc. 8: **8** pt, exc. 9: **9** pt, exc. 10: **7** pt.

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1. Find the inverse of the matrix $C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

2. It is given that

$$A = \begin{bmatrix} \| & \| & \| & \| & \| \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ \| & \| & \| & \| & \| \end{bmatrix} \sim \begin{bmatrix} \| & \| & \| & \| & \| \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 \\ \| & \| & \| & \| & \| \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 1 & 2 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

That is: these matrices are row equivalent.

a. Give the dimension of the column space of A.

b. Which of the following sets can be taken as a basis for Col A (there may be several):

$\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\}$, $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_5\}$, $\{\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\}$,
$\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_5\}$, $\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$.

3. Suppose $\det A = 3$, $\det B = 2$ and $\det C = 4$ for three 4×4 matrices A, B, C. Insofar as possible find the determinants of the matrices $A(B - C)$, $(ABC)^T$ and $2AB^2C^{-1}$. Give the answer "U" (= "unknown") if a determinant cannot be computed.

4. Find a unit vector that is in the orthogonal complement of $\text{Span}\left\{ \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$.

5. **a.** Find all (also the complex) eigenvalues of the matrix $M = \begin{bmatrix} 1 & 2 & -2 \\ 0 & -2 & 0 \\ 2 & 2 & 1 \end{bmatrix}$.

b. Give an eigenvector for one of the complex eigenvalues.

6. Let H be the subspace of $\mathbb{R}^4$ generated by the vector $\begin{bmatrix} 2 \\ 1 \\ -2 \\ 2 \end{bmatrix}$.

Write the vector $\begin{bmatrix} 4 \\ -9 \\ 10 \\ 4 \end{bmatrix}$ as the sum of a vector $\mathbf{v}$ in H and a vector $\mathbf{w}$ in $H^{\perp}$.

7. Consider the (over determined) linear system $\begin{cases} x_1 + 2x_2 = 1 \\ x_1 + x_2 = 8 \\ 2x_1 - x_2 = 10 \\ 3x_1 + x_2 = 9 \end{cases}$

a. Give the augmented matrix of the normal equations that you have to solve to find the least-squares solution of this system.

b. Find the least-squares solution.

8. For each statement indicate whether it is true or false and give a short argument (or a counter example) to support your answer.

a. Suppose $AB = 2I$, for two $n \times n$ matrices A and B. Then $BA = 2I$ as well.

b. Suppose $\mathbf{v}$ is an eigenvector of both the matrix A and the matrix B. Then $\mathbf{v}$ will also be an eigenvector of the matrix AB.

c. If $\{\mathbf{a}_1, \mathbf{a}_2\}$ is a linearly independent set, then the vectors $\mathbf{a}_1 + \mathbf{a}_2$ and $\mathbf{a}_1 - \mathbf{a}_2$ are also linearly independent.

d. If $\{\mathbf{a}_1, \mathbf{a}_2\}$ is an orthogonal set, then $\{\mathbf{a}_1 + \mathbf{a}_2,\ \mathbf{a}_1 - \mathbf{a}_2\}$ is also an orthogonal set.

For the last two exercises you have to give **complete solutions.**
Answer the parts in the correct order!!
Use the first half of a double A4 sheet for exercise 9 and the other half for exc 10.

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9. It is given that $E = \begin{bmatrix} 1 & 0 & -1 & -2 & 1 \\ 2 & 1 & -5 & -2 & 1 \\ -2 & 3 & -7 & 11 & -3 \\ 1 & 1 & -4 & 3 & 6 \end{bmatrix}$ and $F = \begin{bmatrix} 1 & 0 & -1 & -2 & 1 \\ 0 & 1 & -3 & 2 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

are row equivalent. Furthermore, put $A = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 2 & 1 & -5 & -2 \\ -2 & 3 & -7 & 11 \\ 1 & 1 & -4 & 3 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ -3 \\ 6 \end{bmatrix}$.

(Note that $E = [\,A\ \mathbf{y}\,]$.)

a. Find a basis for the column space of A. (Explain **what** you do, and **why**.)

b. Find a basis for the null space of A.

c. Check whether the vector $\mathbf{r} = [\,0\ 0\ 0\ 1\,]$ is in the row space of A.

Now consider the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ with standard matrix A.

d. Check whether $\mathbf{y}$ is in the range of T.

e. Complete the definition: a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if ....

f. Starting from this definition, check whether this transformation T is onto.

10. Let A be given by $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}$, and $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Note that A is a symmetric matrix, so it must be orthogonally diagonalizable.

a. Complete the definition: a matrix A is orthogonally diagonalizable if ....

b. Show that $\mathbf{v}_1$ is an eigenvector of A.

c. Find all eigenvalues of A, and for each eigenvalue find a basis for the corresponding eigenspace.
Hint: for which value of λ will the matrix $(A - \lambda I)$ clearly have determinant 0?

d. Find matrices P and D that orthogonally diagonalize A.

SOLUTIONS

1 $C^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

2a $\dim \operatorname{Col} A = 4$.

2b From **a.** it follows that the column space is ('the whole') $\mathbb{R}^4$. So any set of four independent vectors is a basis for Col A.
So all except $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_5\}$, and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_5\}$ are okay.

3 **U**, 24, $2^4 \cdot 3 \cdot 2^2 \cdot \frac{1}{4} = 48$ respectively.

4 $\pm\frac{1}{7}\begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix}$.

5a $\lambda_1 = -2$; $\lambda_{2,3} = 1 \pm 2i$.

5b For $\lambda = 1 + 2i$: $\mathbf{v} = \begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix}$ does the trick.
You can also take the conjugates of both!

6 $\mathbf{q} = \begin{bmatrix} 4 \\ -9 \\ 10 \\ 4 \end{bmatrix} = \mathbf{v} + \mathbf{w} = (-1)\begin{bmatrix} 2 \\ 1 \\ -2 \\ 2 \end{bmatrix} + \begin{bmatrix} 6 \\ -8 \\ 8 \\ 6 \end{bmatrix}$

7a $\left[\begin{array}{cc|c} 15 & 4 & 56 \\ 4 & 7 & 9 \end{array}\right]$

7b $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$

8a TRUE. Namely, if $AB = I$ for a **square** matrix A, then $B = A^{-1}$, and then also $BA = I$. In this case, from $AB = 2I$, it follows that $A(\frac{1}{2}B) = I$, so $\frac{1}{2}B = A^{-1}$, and it follows that $(\frac{1}{2}B)A = \frac{1}{2}AB = I$. Multiplying the last equation by 2 gives the result $AB = 2I$.

8b TRUE, Simple!!, $A\mathbf{v} = \lambda_1\mathbf{v}$ and $B\mathbf{v} = \lambda_2\mathbf{v}$ imply that $AB\mathbf{v} = A\lambda_2\mathbf{v} = \lambda_1\lambda_2\mathbf{v}$, which show that $\mathbf{v}$ is an eigenvector of AB for the eigenvalue $\lambda_1\lambda_2$.

8c TRUE: Suppose $c_1(\mathbf{a}_1 + \mathbf{a}_2) + c_2(\mathbf{a}_1 - \mathbf{a}_2) = \mathbf{0}$.
Reordering terms gives $(c_1 + c_2)\mathbf{a}_1 + (c_1 - c_2)\mathbf{a}_2 = \mathbf{0}$, and from the linear independence of $\{\mathbf{a}_1, \mathbf{a}_2\}$ we may conclude that $(c_1+c_2) = 0$ and also $(c_1-c_2) = 0$. This implies $c_1 = c_2 = 0$, so that there is no non-trivial combination $c_1(\mathbf{a}_1 + \mathbf{a}_2) + c_2(\mathbf{a}_1 - \mathbf{a}_2)$ that is equal to $\mathbf{0}$.

8d FALSE: $(\mathbf{a}_1 + \mathbf{a}_2) \cdot (\mathbf{a}_1 - \mathbf{a}_2) = \mathbf{a}_1 \cdot \mathbf{a}_1 - \mathbf{a}_2 \cdot \mathbf{a}_2$,
and this is only equal to zero if $||\mathbf{a}_1|| = ||\mathbf{a}_2||$.
So in general the statement is FALSE.

9a Row reduction does not affect the relations between the columns. Row reducing A leads to F_1, the matrix with the first four coumns of F. Here the 1st,2nd and 4th column give a maximal set of independent columns, i.e. a basis for $\mathrm{Col}\, F_1$. Then the corresponding columns of A give a basis for $\mathrm{Col}\, A$. So (for instance) $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$ is a basis for $\mathrm{Col}\, A$.

9b Row reduction does not change null space, i.e. solutions of $A\mathbf{x} = \mathbf{0}$. Put otherwise: $\mathrm{Nul}\, A = \mathrm{Nul}\, F_1$, and with just one extra step a basis for the last subspace is quickly found:

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here the third 'variable' can be taken as a free variable, which lead to the basis $\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix} \right\}$.

9c By row reduction the row space doesn't change, and from the equivalence of E and F it can be immediately seen that $A \sim \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, where $\mathbf{r}$ is just the third row.

So **yes** $\mathbf{r} \in \mathrm{row}\, A$.

9d Since $[\,A\,|\,\mathbf{y}\,] \sim F$ it follows that the equation $T(\mathbf{x}) = \mathbf{y}$, which is the same as $A\mathbf{x} = \mathbf{y}$ is consistent. So **yes** again, $\mathbf{y} \in \operatorname{Col} A = \operatorname{Range}(T)$.

9e Definition of an onto transformation ....

9f T is **not** onto. Since A has only three pivot positions, not every system with augmented matric $[\,A\,|\,\mathbf{b}\,]$ will be consistent.
Other argument: A has only three independent columns, so the range of T, which is the column space of A, is a three-dimensional subspace of $\mathbb{R}^4$, and cannot be equal to the whole $\mathbb{R}^4$.

10a Definition of orthogonally diagonalizable ....

10b It is easily seen that $A\mathbf{v}_1 = \begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix} = 7\mathbf{v}_1$,
so $\mathbf{v}_1$ is an eigenvector of A for the eigenvalue $\lambda_1 = 7$.

10c Following the hint: $A - 1\,I = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$ has clearly (very) dependent columns, and it is also quickly seen that the null space of $(A - 1\,I)$ has dimension 2,
with (for instance) the basis $\{\mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$

10d To diagonalize A we can use $D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $P_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$.
To 'orthogonalize' , we need an orthonormal basis for the eigenspace for $\lambda_{2,3} = 1$.
We can use Gram-Schmidt to fix this:

$$\mathbf{u}_2 = \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

Normalizing these vectors and putting them in a matrix gives

one of the (many) possible answer(s): $P = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \end{bmatrix}$.

Credits

1.	2a	2b	3.	4.	5a	5b	6.	7a	7b
2	2	2	3	2	2	2	2	2	1
8a	**8b**	**8c**	**8d**	**9a**	**9b**	**9c**	**9d**	**9e**	**9f**
2	2	2	2	2	2	1	2	1	1
10a	**10b**	**10c**	**10d**						
1	1	2	3						