Final Exam: Linear Algebra CSE1205
April 17, 2019, 13:30 – 16:30 hs

- Calculators and formula sheets are **not** allowed.
- Credits: 2 points for questions from Part I (except question 17 and 18; 1 point for these) and 5 points for questions from Part II.
- The final score: Sum and divide by 6.

PART I: MULTIPLE CHOICE QUESTIONS

1. How many solutions does the following system of equations has?

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ 2x_1 + x_2 + 3x_3 &= 10 \\ x_1 + 3x_2 + 2x_3 &= 13 \end{aligned}$$

A. No solution **B.** ∞ many solutions
C. A unique solution **D.** None of the other statements apply

Answer: C.

The system is consistent, with unique solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$.

2. Consider the vectors $\mathbf{b}_1 = \begin{bmatrix} 1 & 1 & -1 & 2 \end{bmatrix}^T$, $\mathbf{b}_2 = \begin{bmatrix} -1 & 3 & 0 & 1 \end{bmatrix}^T$, $\mathbf{b}_3 = \begin{bmatrix} 3 & -1 & -2 & 3 \end{bmatrix}^T$. The dimension of $\text{Span}\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is:

A. 0 **B.** 1 **C.** 2 **D.** 3 **E.** 4

Answer: C.

The matrix $A = [\mathbf{b}_1\, \mathbf{b}_2\, \mathbf{b}_3]$ has rank 2 (meaning: it has 2 pivot positions), so the dimension is 2.

3. It is given that $AB = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix}$, where $B = \begin{bmatrix} 4 & 5 \\ 1 & 1 \end{bmatrix}$. The (3,2)-entry a_{32} of A is equal to:

A. -17 **B.** -13 **C.** -7 **D.** -1 **E.** 1 **F.** 7 **G.** 13 **H.** 17

Answer: B.

We have $A = (AB)B^{-1} = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} -1 & 5 \\ 1 & -4 \end{bmatrix}$. The $(3,2)$-entry a_{32} of A is therefore equal to $3 \cdot 5 + 7 \cdot (-4) = -13$.

4. Suppose the equation $A\mathbf{x} = \mathbf{b}$, for an $n \times n$ matrix A, is inconsistent for *some* $\mathbf{b}$ in $\mathbb{R}^n$. Which of the following statements *must* be true?

A. $\det A \neq 0$
B. The columns of A are linearly independent
C. $\operatorname{Col} A = \mathbb{R}^n$
D. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
E. A has n pivot positions
F. The map $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one
G. $\operatorname{Nul} A \neq \{\mathbf{0}\}$
H. None of the others

Answer: G.

Because of the Invertible Matrix Theorem: $\operatorname{Nul} A \neq \{\mathbf{0}\}$

5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that leaves $\mathbf{e}_1$ unchanged, while $\mathbf{e}_2$ is mapped to $-2\mathbf{e}_1 + \mathbf{e}_2$. Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that reflects points though the line $y = -x$. Find the standard matrix for the composition $S \circ T$:

A. $\begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}$ **B.** $\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$ **C.** $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ **D.** $\begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}$

E. $\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$ **F.** $\begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}$ **G.** $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ **H.** $\begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}$

Answer: D.

The standard matrix of $S \circ T$ is given by

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}$$

6. Which of the sets $W_1 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \,\middle|\, x + y + z = 2 \right\}$, $W_2 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \,\middle|\, 2x - y = z,\ x = y \right\}$ is a subspace of $\mathbb{R}^3$?

A. None of them **B.** Only W_1 **C.** Only W_2 **D.** Both

Answer: C.

Only W_2 is a subspace, it is a plane passing through the origin.

W_1 is an affine line, so it is not a subspace.

7. The dimension of $\operatorname{Nul}(A)$, where $A = \begin{bmatrix} -2 & 2 & 3 & 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 3 & 2 & 0 & 1 \end{bmatrix}$, equals:

A. 0 **B.** 1 **C.** 2 **D.** 3 **E.** 4 **F.** 5 **G.** 6 **H.** 7

Answer: E.

The rank of A is 3 (since it has 3 pivot positions), so by the rank theorem the dimension of $\operatorname{Nul}(A)$ is $7 - 3 = 4$.

8. Let A and B be two invertible matrices, $\mathbf{v}$ a non-zero vector and λ a non-zero scalar such that $A\mathbf{v} = \lambda B\mathbf{v}$. Then:

A. λ^{-1} is an eigenvalue of $B^{-1}A$
B. λ is an eigenvalue of $A^{-1}B$
C. λ^{-1} is an eigenvalue of BA^{-1}
D. λ is an eigenvalue of AB^{-1}

E. λ^{-1} is an eigenvalue of $A^{-1}B$ **F.** λ is an eigenvalue of BA^{-1}
G. λ^{-1} is an eigenvalue of AB^{-1} **H.** None of the other options

Answer: E.

Multiplying both sides of $A\mathbf{v} = \lambda B\mathbf{v}$ by A^{-1} from the left and then dividing both sides by λ shows that λ^{-1} is an eigenvalue $A^{-1}B$. Observe that $\lambda \neq 0$ since A is invertible.

9. Calculate A^5, where $A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$:

A. $\begin{bmatrix} -30 & -62 \\ 31 & 63 \end{bmatrix}$ **B.** $\begin{bmatrix} 0 & 1 \\ -2 & 243 \end{bmatrix}$ **C.** $\begin{bmatrix} 63 & -31 \\ -62 & -30 \end{bmatrix}$ **D.** $\begin{bmatrix} -30 & 63 \\ -62 & 31 \end{bmatrix}$
E. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **F.** $\begin{bmatrix} -62 & 63 \\ -30 & 31 \end{bmatrix}$ **G.** $\begin{bmatrix} 0 & 1 \\ -32 & 243 \end{bmatrix}$ **H.** $\begin{bmatrix} -30 & 31 \\ -62 & 63 \end{bmatrix}$

Answer: H.

The matrix A is diagonalizable: $A = PDP^{-1}$, where $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

Whence $A^5 = PD^5P^{-1} = \begin{bmatrix} -30 & 31 \\ -62 & 63 \end{bmatrix}$.

10. Consider the vectors $\mathbf{b}_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$, $\mathbf{b}_2 = \begin{bmatrix} 4 & 0 & 0 & 0 \end{bmatrix}^T$ and $\mathbf{b}_3 = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}^T$. If we apply the Gram-Schmidt process to $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ to obtain an orthogonal set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, then $\mathbf{v}_3$ is (up to rescaling) equal to

A. $\begin{bmatrix} 0 & -1 & 1 & 0 \end{bmatrix}^T$ **B.** $\begin{bmatrix} 0 & -2 & 1 & 1 \end{bmatrix}^T$
C. $\begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}^T$ **D.** $\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$
E. $\begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix}^T$ **F.** $\begin{bmatrix} 0 & 0 & 1 & -1 \end{bmatrix}^T$
G. $\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}^T$ **H.** $\begin{bmatrix} 0 & -1 & 0 & 1 \end{bmatrix}^T$

Answer: B.

Applying Gram-Schmidt yields $\mathbf{v}_2 = \mathbf{b}_2 - \mathbf{b}_1 = \begin{bmatrix} 3 \\ -1 \\ -1 \\ -1 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}$

11. Determine all the distinct (real and complex) eigenvalues of the matrix
$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & 4 & -1 \end{bmatrix}$.

A. $1, -1+2i, -1-2i$ **B.** 1 **C.** $1, 1+2i, 1-2i$
D. $1, 1+2i, -1-2i$ **E.** $1, -2+i, -2-i$ **F.** $1, 1+i, 1-i$
G. $1, 3, -1$ **H.** $1, 2+i, 2-i$

Answer: C.

The characteristic polynomial of A is given by $p(\lambda) = (1-\lambda)(\lambda^2 - 2\lambda + 5)$. Using the quadratic formula we conclude that the roots are given by $1, 1 \pm 2i$

12. Calculate the inverse of the matrix $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix}$, if it exists.

Then the sum of all the entries of A^{-1} is equal to:

A. -3 **B.** -2 **C.** -1 **D.** 0 **E.** 1 **F.** 2 **G.** 3 **H.** A is not invertible

Answer: F.

We have $A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{3}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$, so the sum of the entries is 2.

13. Find the distance from $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ to $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$:

A. 1 **B.** 2 **C.** 5 **D.** $\sqrt{29}$ **E.** $\sqrt{30}$ **F.** 10 **G.** 29 **H.** 30

Answer: A.

The projection of $\mathbf{y}$ onto W is given by $\hat{\mathbf{y}} = \begin{bmatrix} 3/2 \\ 3/2 \\ 7/2 \\ 7/2 \end{bmatrix}$ and the distance is equal to

$$||\mathbf{y} - \hat{\mathbf{y}}|| = \left|\left| \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right|\right| = \sqrt{1} = 1$$

For questions **14** and **15** consider the matrix $A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$.

14. The algebraic multiplicity of the eigenvalue 2 of the above matrix A equals

A. 0 **B.** 1 **C.** 2 **D.** 3 **E.** 4 **F.** 5

Answer: E.

The characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ is given by $(1-\lambda)(2-\lambda)^4$, so the algebraic multiplicity is 4.

15. The geometric multiplicity of the eigenvalue 2 of the above matrix A equals

A. 0 **B.** 1 **C.** 2 **D.** 3 **E.** 4 **F.** 5

Answer: D.

The matrix $A - 2I = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \boxed{-1} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{-2} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ has 2 pivot positions, therefore $\dim E_2 = \dim \text{Nul}(A - 2I) = 5 - 2 = 3$.

16. Suppose the equation $A(XB^{-1})^T = C$ holds for invertible matrices A, B, C. Solving for X gives that X is equal to:

A. $C(A^{-1})^TB$ **B.** $A^{-1}CB$ **C.** $(A^{-1})^TC^TB$ **D.** $C^TA^{-1}B$
E. $(A^{-1})^TC^TB^T$ **F.** $C^T(A^{-1})^TB$ **G.** $(A^{-1})^TCB^T$ **H.** None of the above

Answer: F.

Solving for X yields $X = C^T(A^{-1})^TB$

17. Suppose that the square matrix A is row equivalent to B and $\lambda = 1$ is an eigenvalue of A. Is $\lambda = 1$ also an eigenvalue of B?

A. True **B.** False

Answer: B.

This is in general false (if $\lambda \neq 0$). Example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

18. Suppose that the square matrix A is row equivalent to B and $\lambda = 0$ is an eigenvalue of A. Is $\lambda = 0$ also an eigenvalue of B?

A. True **B.** False

Answer: A.

By the Invertible Matrix Theorem: $\lambda = 0$ is an eigenvalue of $A \iff A$ is singular (i.e. not invertible). Since B is row equivalent to A, it follows that B is also singular. Therefore $\lambda = 0$ is also an eigenvalue of B.

19. Which of the following statements are always true if Q is a (not necessarily square) matrix with **orthonormal rows**?

(I) $Q^TQ = I$ (II) $QQ^T = I$

A. Both are false **B.** Only (I) is true **C.** Only (II) is true **D.** Both are true

Answer: C.

A matrix Q has orthonormal columns if and only if $Q^TQ = I$. Since Q has orthonormal rows if and only if Q^T has orthonormal columns, it follows that Q has orthonormal rows if and only if $(Q^T)^T(Q^T) = I$, that is $QQ^T = I$.

20. Consider the following statements for a square matrix A.

(I) If A invertible and diagonalizable, then A^{-1} is also diagonalizable

(II) If A diagonalizable, then A^T is also diagonalizable

A. Both statements are false **B.** Only (I) is true
C. Only (II) is true **D.** Both statements are true

Answer: D.

Suppose that A is diagonalizable: $A = PDP^{-1}$, with D a diagonal matrix.

The matrix A^T is also diagonalizable since $A^T = (P^T)^{-1}D^TP^T = (P^T)^{-1}DP^T$.

If A is also invertible, then A^{-1} is also diagonalizable because $A^{-1} = PD^{-1}P^{-1}$.

21. Find the equation $y = \beta_0 + \beta_1 x$ of the best line (in the least-squares sense) that fits the points $(0, 1), (2, 3), (4, 2)$.

A. $y = 2$ **B.** $y = 0.5 + x$ **C.** $y = 1 + x$ **D.** $y = 1.5 + 0.25x$
E. $y = 1 + 0.25x$ **F.** $y = 4 - 0.5x$ **G.** $y = 1$ **H.** $y = 1 + 0.5x$

Answer: D.

The vector $\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$ is a least-square solution of the system $X \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \mathbf{y}$, where X is the design matrix and $\mathbf{y}$ the observation vector of the data:

$$X = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

The corresponding normal equations are given by $X^T X \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = X^T \mathbf{y}$, i.e.

$$\begin{bmatrix} 3 & 6 \\ 6 & 20 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

This system has the unique solution $\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 0.25 \end{bmatrix}$

END OF PART I.
GO TO PART II: TRUE/FALSE QUESTIONS

CSE1205 (Linear Algebra), 17 – 04 – 2019, True/False Questions

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*write **readable** and underline your surname*	

You are asked to decide whether the statements are true or false.
Either give a proof or a specific counterexample.

22. If A and B are matrices such that AB exists, then: if $\mathbf{x} \in \text{Col}(AB) \implies \mathbf{x} \in \text{Col}(A)$.

Answer: TRUE.

Solution 1:

If $\mathbf{x} \in \text{Col}(AB)$, then $\mathbf{x}$ is a linear combination of the columns of AB. This means that there exists a certain vector $\mathbf{c}$ such that $\mathbf{x} = AB\mathbf{c}$.

Therefore $\mathbf{x} = A\mathbf{c}'$ where $\mathbf{c}' = B\mathbf{c}$.

This proves that $\mathbf{x} \in \text{Col}(A)$.

Solution 2:

If $B = [\mathbf{b}_1 \cdots \mathbf{b}_n]$, then

$$\text{Col}(AB) = \text{Span}\{A\mathbf{b}_1 \ \ldots \ A\mathbf{b}_n\}$$

But $A\mathbf{b}_j \in \text{Col}(A)$ for any j, and therefore: $\text{Col}(AB) \subseteq \text{Col}(A)$

23. If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ are linearly independent vectors, then $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_3 + \mathbf{v}_4, 2\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_4\}$ are also linearly independent.

Answer: FALSE.

A set of vectors $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is linearly independent if the vector equation

$$x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n = \mathbf{0}$$

admits only the trivial solution.

So we need to check whether the following system has non-trivial solutions or not:

$$x_1(\mathbf{v}_1 + \mathbf{v}_2) + x_2(\mathbf{v}_2 + \mathbf{v}_3) + x_3(\mathbf{v}_3 + \mathbf{v}_4) + x_4(2\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_4) = \mathbf{0}. \quad (*)$$

We can re-write $(*)$ as follows:

$$(x_1 + 2x_4)\mathbf{v}_1 + (x_1 + x_2 + x_4)\mathbf{v}_2 + (x_2 + x_3)\mathbf{v}_3 + (x_3 + x_4)\mathbf{v}_4 = \mathbf{0},$$

By the linear independence of the $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ this is equivalent to the system

$$\begin{cases} x_1 + 2x_4 = 0 \\ x_1 + x_2 + x_4 = 0 \\ x_2 + x_3 = 0 \\ x_3 + x_4 = 0 \end{cases}$$

The vectors $\{\mathbf{v}_1+\mathbf{v}_2, \mathbf{v}_2+\mathbf{v}_3, \mathbf{v}_3+\mathbf{v}_4, 2\mathbf{v}_1+\mathbf{v}_2+\mathbf{v}_4\}$ are linearly dependent since thie above system has non-trivial solutions:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \text{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

A concrete counterexample: $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ in $\mathbb{R}^4$.

24. If $A^T A$ is a diagonal matrix, then the columns of A are orthogonal.

Answer: TRUE.

If $A = [\mathbf{a}_1\, \mathbf{a}_2\, \cdots \mathbf{a}_n]$, then

$$A^T A = \begin{bmatrix} -\mathbf{a}_1^T- \\ -\mathbf{a}_1^T- \\ \vdots \\ -\mathbf{a}_n^T- \end{bmatrix} [\mathbf{a}_1\, \mathbf{a}_2\, \cdots \mathbf{a}_n] = \begin{bmatrix} \mathbf{a}_1\cdot\mathbf{a}_1 & \mathbf{a}_1\cdot\mathbf{a}_2 & \cdots & \mathbf{a}_1\cdot\mathbf{a}_n \\ \mathbf{a}_2\cdot\mathbf{a}_1 & \mathbf{a}_2\cdot\mathbf{a}_2 & \cdots & \mathbf{a}_2\cdot\mathbf{a}_n \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_n\cdot\mathbf{a}_1 & \mathbf{a}_n\cdot\mathbf{a}_2 & \cdots & \mathbf{a}_n\cdot\mathbf{a}_n \end{bmatrix} = [\mathbf{a}_i\cdot\mathbf{a}_j]_{i,j}$$

In other words: the (i,j)-the entry of $A^T A$ is given by the inner product $\mathbf{a}_i \cdot \mathbf{a}_j$.

Therefore, if $A^T A = D$, with D a diagonal matrix D with (say) entries $d_1, d_2, \ldots, d_n$ on the diagonal, then

$$\mathbf{a}_i \cdot \mathbf{a}_j = \begin{cases} d_j & \text{if } j = i \\ 0 & \text{if } j \neq j \end{cases}$$

25. If a square matrix A satisfies the equation $2A^2 + 3A = 4I$, then A is invertible.

Answer: TRUE.

Solution 1:
If $2A^2 + 3A = 4I$, then $\frac{1}{2}A^2 + \frac{3}{2}A = I$, and therefore also $A(\frac{1}{2}A + \frac{3}{4}I) = I$ and $(\frac{1}{2}A + \frac{3}{4}I)A = I$.

I.e. $AB = BA = I$ where $B = (\frac{1}{2}A + \frac{3}{4}I)$.

This proves that A is invertible (and furthermore $A^{-1} = B = \frac{1}{2}A + \frac{3}{4}I$).

Solution 2 (with determinants): $AB = I \implies \det A \det B = 1 \implies \det A \neq 0 \implies A$ is invertible by the Invertible Matrix Theorem.

Solution 3 (with determinants): $A(2A+3I) = 4I \implies \det A \det(2A+3I) = 4^n$, where n is the size of A. Therefore $\det A \neq 0$, which implies that A is invertible (again by the

Invertible Matrix Theorem.).

<u>Solution 4 (with eigenvalues):</u>

Suppose that λ is an eigenvalue of A: $A\mathbf{x} = \lambda\mathbf{x}$, with $\mathbf{x} \neq 0$. Together with $2A^2 + 3A = 4I$ we see that $(2\lambda^2 + 3\lambda)\mathbf{x} = 4\mathbf{x}$, and therefore $\lambda(2\lambda + 3) = 4$. This implies that $\lambda \neq 0$, i.e that 0 is not an eigenvalue of A. By the Invertible Matrix Theorem it follows that A is invertible.