

Resit *Linear Algebra* (AESB1210-15), July 2, 2019, 9:00-12:00

- Calculators and formula sheets are **not** allowed.
 - Credits: **4 points** for questions from Part I and **5 points** for questions from Part II.
 - The final score: (Total+5)/6, rounded to 1 decimal.
 - Write your answers on exam paper.
-

PART I: SHORT-ANSWER QUESTIONS

1. Solve the following system of equations:

$$\begin{aligned}x_1 + x_2 + 2x_3 &= 6 \\3x_1 + 2x_2 + 14x_3 &= 5 \\x_1 + 5x_3 &= -2\end{aligned}$$

..... **Answer**

The unique solution is given by $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}$.

2. A is a 3×3 -matrix that satisfies the property $A^2 = 2A^T$. What are the possible values for the determinant of A ?

..... **Answer**

The equation implies that $\det A^2 = 8 \det A$, whence $\det A$ can only be 0 or 8.

3. Let $A = \begin{bmatrix} 4 & 7 & 3 \\ -6 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 6 \\ -5 & 3 & -1 \\ 0 & -4 & -7 \end{bmatrix}$.

- Write the entries in the first row of AB :
- Find the entries in the second column of AB :

..... **Answer**

a. The first row is given by $[-27 \ 5 \ -4]$.

b. The second column is given by $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

4. Determine all possible values of h such that the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} -1 \\ h \\ 1 \end{bmatrix}$ are linearly independent. (Write **NOT** if there are no such h):

..... **Answer**

Condition on h is: $h \neq -4$.

5. Find the LU -decomposition $A = LU$ of the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ -3 & -10 & 2 \end{bmatrix}$:

..... **Answer**

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 4 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

6. For the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ it is given that $T(\mathbf{e}_1 + \mathbf{e}_2) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and

$T(\mathbf{e}_1 - \mathbf{e}_2) = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$. Find the standard matrix A of T :

..... **Answer**

The standard matrix is given by $A = \begin{bmatrix} 4 & -3 \\ 5 & -3 \\ 6 & -3 \end{bmatrix}$

7. Find the orthogonal projection of $\mathbf{y} = \begin{bmatrix} 18 \\ 12 \\ 6 \end{bmatrix}$ onto $W = \text{Span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$:

..... **Answer**

The projection is given by $\hat{\mathbf{y}} = \begin{bmatrix} 8 \\ 17 \\ 1 \end{bmatrix}$

8. Consider the matrix $A = \begin{bmatrix} 3 & -4 & 7 \\ 4 & 3 & -2 \\ 0 & 0 & 1 \end{bmatrix}$.

a. Find the characteristic polynomial $p(\lambda)$ of A :

b. Find all the (real, and possibly, complex) eigenvalues of A :

..... **Answer**

a. $p(\lambda) = \det(A - \lambda I)$ and therefore $p(\lambda) = (1 - \lambda)(\lambda^2 - 6\lambda + 25)$

b. The eigenvalues are given by 1, $3 + 4i$ and $3 - 4i$.

9. Consider the overdetermined system $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} -1 & 1 \\ 2 & -1 \\ 0 & -1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$.

a. Give the augmented matrix of the normal equations that you have to solve to find the least-squares solution of this system:

b. Find all the least-squares solutions:

..... **Answer**

a. The corresponding normal equations are given by $A^T A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^T \mathbf{b}$.

Explicitly:

$$\begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -6 \end{bmatrix}$$

b. This system has the unique solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -5/2 \end{bmatrix}$.

10. a. Find the dimension of $W := \text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \\ 0 \end{bmatrix} \right\}$ in \mathbb{R}^4 :

b. Find a basis for W^\perp :

..... **Answer**

a. The dimension of W is 2.

b. A basis is given by (for example): $\begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

END OF PART I. GO TO PART II (OPEN QUESTIONS)!

(continued on next page!)

PART II: OPEN QUESTIONS

- Justify clearly your answer.
- Mention the theorems, corollaries and results you are using!

11. a. Find the reduced echelon form of the matrix $A := \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$
- b. Find the rank of A .
- c. Find the dimension of $\text{Nul}(A)$.
- d. Find a basis for $\text{Col}(A)$.

..... **Answer**

- a. Applying row operations yields that $U = \begin{bmatrix} 1 & 0 & -5 & -5 & 0 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

- b. The rank is equal to the number of pivots, so it equals 2.
- c. By the Rank Theorem $\dim \text{Nul}A$ is equal to $3 = 5 - 2$.
- d. The first and second column of U contain a pivot, therefore the first and second columns of A form a basis for $\text{Col}A$.

(continued on next page!)

12. a. Calculate the determinant of the matrix $A := \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & a \end{bmatrix}$.
b. Find all values of a such that A is invertible.
c. Find the inverse A^{-1} for $a = 1$.

..... **Answer**

- a. Expanding along rows/columns or bringing A in echelon form yields that

$$\det A = a$$

- b. A invertible if and only if $\det A \neq 0$, that is, if and only if $a \neq 0$.

- c. The reduced echelon form of $[A|I]$ is given by $[I|A^{-1}]$.

Bringing $[A|I]$ in reduced echelon form, using elementary row operations, yields that

$$A^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}.$$

(continued on next page!)

13. a. Find the eigenvalues of $A := \begin{bmatrix} -1 & 4 \\ 2 & -3 \end{bmatrix}$.
- b. Find a basis for all the eigenspaces E_λ .
- c. Give a diagonalization of A (if it exists), i.e. write A in the form PDP^{-1} with D a diagonal matrix.

..... **Answer**

- a. The characteristic equation is given by $\det(A - \lambda I) = 0$, i.e. $(\lambda + 5)(\lambda - 1) = 0$. Therefore the eigenvalues are given by -5 and 1 .
- b. By calculating $\text{Nul}(A + 5I)$ we see that the eigenspace corresponding to $\lambda = -5$ is spanned by $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
By calculating $\text{Nul}(A - I)$ we see that the eigenspace corresponding to $\lambda = 1$ is spanned by $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.
- c. In class we have seen that A diagonalizable if and only A has a basis of eigenvectors, so by part **b.** A is indeed diagonalizable. The theory also tells us that we can take:

$$P = [\mathbf{v}_1 \mathbf{v}_2] = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -5 & 0 \\ 0 & 1 \end{bmatrix}$$