Resit *Linear Algebra* (AESB1210-15), July 2, 2019, 9:00-12:00

- Calculators and formula sheets are **not** allowed.
- Credits: **4 points** for questions from Part I and **5 points** for questions from Part II.
- The final score: (Total+5)/6, rounded to 1 decimal.
- Write your answers on exam paper.

PART I: SHORT-ANSWER QUESTIONS

1. Solve the following system of equations:

$$\begin{aligned} x_1 + \ \ x_2 + \ \ 2x_3 &= \ \ 6 \\ 3x_1 + 2x_2 + 14x_3 &= \ \ 5 \\ x_1 \qquad \ + \ \ 5x_3 &= -2 \end{aligned}$$

.... **Answer** ....

The unique solution is given by $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}$.

2. A is a 3×3-matrix that satisfies the property $A^2 = 2A^T$. What are the possible values for the determinant of A?

.... **Answer** ....

The equation implies that $\det A^2 = 8 \det A$, whence $\det A$ can only be 0 or 8.

3. Let $A = \begin{bmatrix} 4 & 7 & 3 \\ -6 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 6 \\ -5 & 3 & -1 \\ 0 & -4 & -7 \end{bmatrix}$.

a. Write the entries in the first row of AB:

b. Find the entries in the second column of AB:

Answer

a. The first row is given by $\begin{bmatrix}-27 & 5 & -4\end{bmatrix}$.

b. The second column is given by $\begin{bmatrix}5\\1\end{bmatrix}$.

4. Determine all possible values of h such that the vectors $\mathbf{v}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}1\\-2\\1\end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix}-1\\h\\1\end{bmatrix}$ are linearly independent. (Write **NOT** if there are no such h):

Answer

Condition on h is: $h \neq -4$.

5. Find the LU-decomposition $A = LU$ of the matrix $A = \begin{bmatrix}1 & 2 & 1\\2 & 3 & 3\\-3 & -10 & 2\end{bmatrix}$:

Answer

$$L = \begin{bmatrix}1 & 0 & 0\\2 & 1 & 0\\-3 & 4 & 1\end{bmatrix} \qquad U = \begin{bmatrix}1 & 2 & 1\\0 & -1 & 1\\0 & 0 & 1\end{bmatrix}$$

6. For the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ it is given that $T(\mathbf{e}_1 + \mathbf{e}_2) = \begin{bmatrix}1\\2\\3\end{bmatrix}$ and $T(\mathbf{e}_1 - \mathbf{e}_2) = \begin{bmatrix}7\\8\\9\end{bmatrix}$. Find the standard matrix A of T:

Answer

The standard matrix is given by $A = \begin{bmatrix}4 & -3\\5 & -3\\6 & -3\end{bmatrix}$

7. Find the orthogonal projection of $\mathbf{y} = \begin{bmatrix}18\\12\\6\end{bmatrix}$ onto $W = \text{Span}\left\{\begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}\right\}$:

Answer

The projection is given by $\hat{\mathbf{y}} = \begin{bmatrix}8\\17\\1\end{bmatrix}$

8. Consider the matrix $A = \begin{bmatrix}3 & -4 & 7\\4 & 3 & -2\\0 & 0 & 1\end{bmatrix}$.

a. Find the characteristic polynomial $p(\lambda)$ of A:

b. Find all the (real, and possibly, complex) eigenvalues of A:

.... **Answer** ....

a. $p(\lambda) = \det(A - \lambda I)$ and therefore $p(\lambda) = (1-\lambda)(\lambda^2 - 6\lambda + 25)$

b. The eigenvalues are given by 1, $3+4i$ and $3-4i$. .

9. Consider the overdetermined system $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} -1 & 1 \\ 2 & -1 \\ 0 & -1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$.

a. Give the augmented matrix of the normal equations that you have to solve to find the least-squares solution of this system:

b. Find all the least-squares solutions:

.... **Answer** ....

a. The corresponding normal equations are given by $A^T A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^T\mathbf{b}$.

Explicitly:

$$\begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -6 \end{bmatrix}$$

b. This system has the unique solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -5/2 \end{bmatrix}$.

10. **a.** Find the dimension of $W := \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \\ 0 \end{bmatrix} \right\}$ in $\mathbb{R}^4$:

b. Find a basis for $W^\perp$:

.... **Answer** ....

a. The dimension of W is 2.

b. A basis is given by (for example): $\begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

END OF PART I. GO TO PART II (OPEN QUESTIONS)!

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PART II: OPEN QUESTIONS

- Justify clearly your answer.
- Mention the theorems, corollaries and results you are using!

11. **a.** Find the reduced echelon form of the matrix $A := \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$

b. Find the rank of A.

c. Find the dimension of $\mathrm{Nul}(A)$.

d. Find a basis for $\mathrm{Col}(A)$.

.... **Answer** ....

a. Applying row operations yields that $U = \begin{bmatrix} 1 & 0 & -5 & -5 & 0 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

b. The rank is equal to the number of pivots, so it equals 2.

c. By the Rank Theorem $\dim \mathrm{Nul} A$ is equal to $2 = 5 - 3$.

d. The first and second column of U contain a pivot, therefore the first and second columns of A form a basis for $\mathrm{Col} A$.

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12. **a.** Calculate the determinant of the matrix $A := \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & a \end{bmatrix}$.

b. Find all values of a such that A is invertible.

c. Find the inverse A^{-1} for $a = 1$.

.... **Answer** ....

a. Expanding along rows/columns or bringing A in echelon form yields that

$$\det A = a$$

b. A invertible if and only if $\det A \neq 0$, that is, if and only if $a \neq 0$.

c. The reduced echelon form of $[A|I]$ is given by $[I|A^{-1}]$. .
Bringing $[A|I]$ in reduced echelon form, using elementary row operations, yields that $A^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$.

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13. **a.** Find the eigenvalues of $A := \begin{bmatrix} -1 & 4 \\ 2 & -3 \end{bmatrix}$.

b. Find a basis for all the eigenspaces E_λ.

c. Give a diagonalization of A (if it exists), i.e. write A in the form PDP^{-1} with D a diagonal matrix.

.... **Answer** ....

a. The characteristic equation is given by $\det(A - \lambda I) = 0$, i.e. $(\lambda + 5)(\lambda - 1) = 0$. Therefore the eigenvalues are given by -5 and 1.

b. By calculating $\mathrm{Nul}(A + 5I)$ we see that the eigenspace corresponding to $\lambda = -5$ is spanned by $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

By calculating $\mathrm{Nul}(A - I)$ we see that the eigenspace corresponding to $\lambda = 1$ is spanned by $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

c. In class we have seen that A diagonalizable if and only A has a basis of eigenvectors, so by part **b.** A is indeed diagonalizable. The theory also tells us that we can take:

$$P = [\mathbf{v}_1 \mathbf{v}_2] = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} -5 & 0 \\ 0 & 1 \end{bmatrix}$$