AES1210-15 (Linear Algebra), 15 − 04 − 2019, Final Exam

Name:	**Student ID:**
*write **readable** and underline your <u>surname</u>*	

- Calculators and formula sheets are **not** allowed.
- Credits: 3 points for questions from Part I and 4 points for questions from Part II.
- The final score: (Total+4)/ 5, rounded to 1 decimal.

PART I: SHORT-ANSWER QUESTIONS

1. Solve the following system of equations:

$$\begin{aligned} x_1 - 2x_2 + \ \ x_3 &= 0 \\ x_1 - \ \ x_2 - 3x_3 &= 4 \\ 3x_1 - 4x_2 + \ \ x_3 &= 2 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$$

2. Let $A = \begin{bmatrix} 1 & -3 & 2 & -4 \\ -3 & 9 & -1 & 5 \\ 2 & -6 & 4 & -3 \\ -4 & 12 & 2 & 7 \end{bmatrix}$

a. Find a basis for Col(A): $\left\{ \qquad \right\}$

b. Find a basis for Nul(A): $\left\{ \qquad \right\}$

3. Consider the following linear transformations:

(1) $T : \mathbb{R}^2 \to \mathbb{R}^3$ has standard matrix $\begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 6 & 5 \end{bmatrix}$

(2) $S : \mathbb{R}^3 \to \mathbb{R}^2$ is given by the formula $S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 + x_3 \\ x_2 + x_3 \end{bmatrix}$

a. Determine the standard matrix of S: $\begin{bmatrix} \quad \end{bmatrix}$

b. Determine the standard matrix of $S \circ T$: $\begin{bmatrix} \quad \end{bmatrix}$

4. Let $A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 2 & 0 & 1 & a \\ 1 & 0 & 3 & 2 \\ 2 & -2 & 1 & 4 \end{bmatrix}$, where a is a scalar. Calculate the determinant of A.

$\det A =$

5. Calculate the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & a \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ for all $a \neq 1$: $A^{-1} = \begin{bmatrix} \quad \end{bmatrix}$

6. Consider the transformation $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + 4x_2 \\ -x_1 - x_2 \end{bmatrix}$.

Find the matrix $[T]_{\mathcal{B}}$ of T relative to the basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, where $\mathbf{b}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Answer: $\begin{bmatrix} \quad \end{bmatrix}$

7. Find an orthogonal basis for $W = \text{Span}\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$, where

$$\mathbf{b}_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 2 \end{bmatrix}$$

Answer: $\left\{ \quad \right\}$

8. Let W be the subspace of $\mathbb{R}^3$ spanned by the vectors $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

Consider the vector $\mathbf{y} = \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}$.

a. Write $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$, with $\hat{\mathbf{y}} \in W$ and $\mathbf{z} \in W^{\perp}$:

$\hat{\mathbf{y}} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$ $\qquad$ $\mathbf{z} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$

b. Calculate the distance $dist(\mathbf{y}, W)$ between $\mathbf{y}$ and W.

Answer:

9. Let $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$.

a. Find the (real and possibly complex) eigenvalues of A:

Eigenvalues of A:

b. For every eigenvalue of A you found in part **a**, find an associated (real and complex) eigenvector.

Eigenvectors:

10. Consider the matrix $A = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{2}{3} & b \\ \frac{1}{\sqrt{2}} & a & b \\ 0 & \frac{1}{3} & -4b \end{bmatrix}$.

Determine all scalars a and b such that A is an orthogonal matrix.

Answer:

END OF PART I.
GO TO PART II (OPEN QUESTIONS)!

PART II: OPEN QUESTIONS

Important: **Mention clearly the theorems, corollaries and results you are using!**

11. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subset \mathbb{R}^7$ be a set of linearly independent set of vectors.

Proof that $\{\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_3 - \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_3\}$ is also a linearly independent set.

12. Suppose that A, B, C are square matrices satisfying $ABC = I$.

Prove that B is invertible and express B^{-1} in terms of A and C.

13. Let $A = \begin{bmatrix} 7 & 2 & 1 \\ -4 & 1 & a \\ 0 & 0 & 5 \end{bmatrix}$, where a is a real constant.

a. Determine all the eigenvalues of A and their algebraic multiplicity.

b. Find a basis for the eigenspace E_λ with $\lambda = 5$.

(continued on next page!)

c. For which value(s) of a is the matrix A diagonalizable?

14. Find the equation of the line $y = \beta_0 + \beta_1 x$ that best fits (in the least-square sense) the points $(0,0), (1,0), (2,1), (3,1)$.