AES1210-15 (Linear Algebra), 15 – 04 – 2019, Final Exam

Name: SOLUTION **Student ID:**

*write **readable** and underline your surname*

- Calculators and formula sheets are **not** allowed.
- Credits: 3 points for questions from Part I and 4 points for questions from Part II.
- The final score: (Total+4)/ 5, rounded to 1 decimal.

PART I: SHORT-ANSWER QUESTIONS

1. Solve the following system of equations:

$$\begin{aligned} x_1 - 2x_2 + \ \ x_3 &= 0 \\ x_1 - \ \ x_2 - 3x_3 &= 4 \\ 3x_1 - 4x_2 + \ \ x_3 &= 2 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

2. Let $A = \begin{bmatrix} 1 & -3 & 2 & -4 \\ -3 & 9 & -1 & 5 \\ 2 & -6 & 4 & -3 \\ -4 & 12 & 2 & 7 \end{bmatrix}$

a. Find a basis for Col(A): $\left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ -3 \\ 7 \end{bmatrix} \right\}$

b. Find a basis for Nul(A): $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

3. Consider the following linear transformations:

(1) $T : \mathbb{R}^2 \to \mathbb{R}^3$ has standard matrix $\begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 6 & 5 \end{bmatrix}$

(2) $S : \mathbb{R}^3 \to \mathbb{R}^2$ is given by the formula $S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 + x_3 \\ x_2 + x_3 \end{bmatrix}$

a. Determine the standard matrix of S: $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

b. Determine the standard matrix of $S \circ T$: $\begin{bmatrix} 7 & 4 \\ 7 & 9 \end{bmatrix}$

4. Let $A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 2 & 0 & 1 & a \\ 1 & 0 & 3 & 2 \\ 2 & -2 & 1 & 4 \end{bmatrix}$, where a is a scalar. Calculate the determinant of A.

$\det A =$ $2a + 12$

5. Calculate the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & a \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ for all $a \neq 1$: $A^{-1} =$ $\begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{1-a} & \frac{a}{a-1} & 0 \\ \frac{1}{a-1} & \frac{1}{1-a} & 0 \end{bmatrix}$

6. Consider the transformation $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + 4x_2 \\ -x_1 - x_2 \end{bmatrix}$.

Find the matrix $[T]_{\mathcal{B}}$ of T relative to the basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, where $\mathbf{b}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Answer: $\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$

7. Find an orthogonal basis for $W = \text{Span}\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$, where

$$\mathbf{b}_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 2 \end{bmatrix}$$

Answer: $\left\{ \begin{bmatrix} 3 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix} \right\}$

8. Let W be the subspace of $\mathbb{R}^3$ spanned by the vectors $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

Consider the vector $\mathbf{y} = \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}$.

a. Write $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$, with $\hat{\mathbf{y}} \in W$ and $\mathbf{z} \in W^{\perp}$:

$$\hat{\mathbf{y}} = \begin{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \end{bmatrix} \qquad \mathbf{z} = \begin{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} \end{bmatrix}$$

b. Calculate the distance $dist(\mathbf{y}, W)$ between $\mathbf{y}$ and W.

Answer: $\sqrt{8}$

9. Let $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$.

a. Find the (real and possibly complex) eigenvalues of A:

Eigenvalues of A: $1 + 2i, 1 - 2i$

b. For every eigenvalue of A you found in part **a**, find an associated (real and complex) eigenvector.

Eigenvectors: $\begin{bmatrix} 1+i \\ 2 \end{bmatrix}, \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$

10. Consider the matrix $A = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{2}{3} & b \\ \frac{1}{\sqrt{2}} & a & b \\ 0 & \frac{1}{3} & -4b \end{bmatrix}$.

Determine all scalars a and b such that A is an orthogonal matrix.

Answer: $a = \frac{2}{3}$, $b = \pm\frac{1}{\sqrt{18}}$

END OF PART I.
GO TO PART II (OPEN QUESTIONS)!

PART II: OPEN QUESTIONS

Important: **Mention clearly the theorems, corollaries and results you are using!**

11. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subset \mathbb{R}^7$ be a set of linearly independent set of vectors.

Proof that $\{\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_3 - \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_3\}$ is also a linearly independent set.

Answer:

The vector equation

$$x_1(\mathbf{v}_1 - \mathbf{v}_2) + x_2(\mathbf{v}_3 - \mathbf{v}_2) + x_3(\mathbf{v}_1 + \mathbf{v}_3) = \mathbf{0}$$

is equivalent to

$$(x_1 + x_3)\mathbf{v}_1 + (-x_1 - x_2)\mathbf{v}_2 + (x_2 + x_3)\mathbf{v}_3 = \mathbf{0},$$

and since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent, to the system

$$\begin{cases} x_1 + x_3 = 0 \\ -x_1 - x_2 = 0 \\ x_2 + x_3 = 0 \end{cases}$$

But the only solution to this system is the trivial solution $x_3 = x_2 = x_1 = 0$.

This implies that the vectors $\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_3 - \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_3$ are linearly independent.

12. Suppose that A, B, C are square matrices satisfying $ABC = I$.

Prove that B is invertible and express B^{-1} in terms of A and C.

Answer: Taking determinants:

$$\det A \det B \det C = 1 \implies \det A, \det B, \det C \neq 0$$

The Invertible Matrix Theorem: A, B, C are invertible.

Multiplying the identity $ABC = I$ first from the left by A^{-1} and then from the right by C^{-1} yields

$$B = A^{-1}IC^{-1} = A^{-1}C^{-1},$$

and therefore

$$B^{-1} = (C^{-1})^{-1}(A^{-1})^{-1} = CA$$

13. Let $A = \begin{bmatrix} 7 & 2 & 1 \\ -4 & 1 & a \\ 0 & 0 & 5 \end{bmatrix}$, where a is a real constant.

a. Determine all the eigenvalues of A and their algebraic multiplicity.

Answer:

The characteristic polynomial of A is given by

$$p(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 7-\lambda & 2 & 1 \\ -4 & 1-\lambda & a \\ 0 & 0 & 5-\lambda \end{bmatrix}$$

Expanding along the last row yields:

$$p(\lambda) = (5-\lambda)\det \begin{bmatrix} 7-\lambda & 2 \\ -4 & 1-\lambda \end{bmatrix} = (5-\lambda)(\lambda^2 - 8\lambda + 15) = -(\lambda-3)(\lambda-5)^2$$

The eigenvalues are therefore:
$\lambda_1 = 3$ (with algebraic multiplicity 1), and
$\lambda_2 = 5$ (with algebraic multiplicity 2).

b. Find a basis for the eigenspace E_λ with $\lambda = 5$.

Answer:

$$A - 5I = \begin{bmatrix} 2 & 2 & 1 \\ -4 & -4 & a \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 1 \\ -4 & -4 & a \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 1 \\ 0 & 0 & a+2 \\ 0 & 0 & 0 \end{bmatrix}$$

If $a + 2 \neq 0$, then the above system has 2 pivot positions and
$E_5 = \text{Nul}(A - 5I) = \mathbb{R} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

If $a + 2 = 0$, then the above system has 1 pivot position and
$E_5 = \text{Nul}(A - 5I) = \text{Span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$.

(continued on next page!)

c. For which value(s) of a is the matrix A diagonalizable?

Answer:
A matrix A is diagonalizable if and only if the geometric multiplicity of any eigenvalue is equal to the algebraic multiplicity.
For the given matrix A this always holds for $\lambda = 3$, but for $\lambda = 5$ it only holds if $a = -2$.
Conclusion: A is diagonalizable if and only if $a = -2$.

14. Find the equation of the line $y = \beta_0 + \beta_1 x$ that best fits (in the least-square sense) the points $(0,0), (1,0), (2,1), (3,1)$.

Answer:
The vector $\begin{bmatrix}\beta_0\\\beta_1\end{bmatrix}$ is a least-square solution of the system $X\begin{bmatrix}\beta_0\\\beta_1\end{bmatrix} = \mathbf{y}$, where X is the design matrix and $\mathbf{y}$ the observation vector of the data:

$$X = \begin{bmatrix}1 & 0\\1 & 1\\1 & 2\\1 & 3\end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

The corresponding normal equations are given by $X^TX\begin{bmatrix}\beta_0\\\beta_1\end{bmatrix} = X^T\mathbf{y}$, i.e.

$$\begin{bmatrix}4 & 6\\6 & 14\end{bmatrix}\begin{bmatrix}\beta_0\\\beta_1\end{bmatrix} = \begin{bmatrix}2\\5\end{bmatrix}$$

This system has the unique solution $\begin{bmatrix}\beta_0\\\beta_1\end{bmatrix} = \begin{bmatrix}-0.1\\0.4\end{bmatrix}$.
So the best line is given by the equation $y = 0.4 - 0.1x$.