Answers

Ex. 1 $xy' + \frac{x}{x+1}y = 5x^3 \iff y' + \frac{1}{x+1}y = 5x^2 \quad (x>0)$

So the given DE is Linear and an integrating factor is $e^{\int \frac{1}{x+1}dx} = e^{\ln(x+1)} = x+1$

Multiplying both sides of the DE by $x+1$, we obtain:

put in standard form

$$(x+1)y' + y = 5x^2(x+1) \iff$$

$$((x+1)y)' = 5x^3 + 5x^2 \iff$$

$$(x+1)y = \frac{5}{4}x^4 + \frac{5}{3}x^3 + C \text{ where } C \in \mathbb{R} \iff$$

$$y(x) = 5\left(\frac{x^4}{4(x+1)} + \frac{x^3}{3(x+1)}\right) + \frac{C}{x+1} \text{ where } C \in \mathbb{R}$$

Ex. 2 $\frac{dy}{dx} = \left(\frac{x-1}{x^2}\right)\frac{1}{y^2}$, so the DE is separabel.

Separating variables and integrating, we calculate:

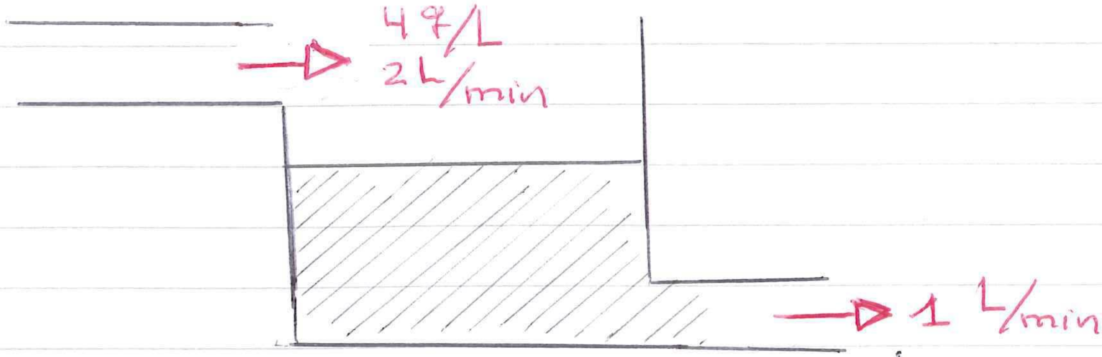
$$\int y^2 \frac{dy}{dx}dx = \int \frac{1}{x} - \frac{1}{x^2}dx \Rightarrow$$

$$\frac{1}{3}y^3 = \ln(x) + \frac{1}{x} + C_1 \quad \text{where } C_1 \in \mathbb{R}$$

$$\Rightarrow y^3 = 3\ln(x) + \frac{3}{x} + C_2 \quad \text{where } C_2 \in \mathbb{R}$$

$$\Rightarrow y(x) = \sqrt[3]{3\ln(x) + \frac{3}{x} + C} \quad \text{where } C \in \mathbb{R}$$

Ex. 3



Define: $y(t)$ is the amount of salt (i g) at time t.
We consider the time interval $[t, t+\Delta t]$ and set up a balance:

$$\text{net change} = \text{inflow} - \text{outflow}$$

$$\Longrightarrow \quad y(t+\Delta t) - y(t) = 4 \cdot 2 \cdot \Delta t - \frac{y(t)}{10+t} \cdot 1 \cdot \Delta t$$

$$\Longrightarrow \quad \frac{y(t+\Delta t) - y(t)}{\Delta t} = 8 - \frac{1}{10+t}\, y(t)$$

Let Δt approach 0, then we find

$$y'(t) = 8 - \frac{1}{10+t}\, y(t) \quad , \text{ so}$$

$$y'(t) + \frac{1}{10+t}\, y(t) = 8$$

Ex. 4 Put $z = 1+i$, then $|z| = \sqrt{2}$, $\arg(z) = \frac{\pi}{4}$

and $(1+i)^5 = z^5 = \sqrt{2}^{\,5}\left(\cos\left(\frac{5}{4}\pi\right) + i\sin\left(\frac{5}{4}\pi\right)\right)$

De Moivre's theorem

$$= 4\sqrt{2}\left(-\frac{1}{2}\sqrt{2} - i\frac{1}{2}\sqrt{2}\right) = -4 - 4i$$

Ex. 5 The fourth roots of -4 are the solutions of the equation $z^4=-4$. So write $z=R(\cos\theta+i\sin\theta)$ and solve

$$z^4=-4$$

$$\Longleftrightarrow R^4\left(\cos(4\theta)+i\sin(4\theta)\right)=4\left(\cos\pi+i\sin\pi\right)$$

de Moivre's theorem

$$\Longleftrightarrow \begin{cases} R^4=4 \Rightarrow R=\sqrt{2} \\ 4\theta=\pi+k2\pi \Rightarrow \theta=\frac{\pi}{4}+k\frac{\pi}{2} \end{cases}$$

$$\Longrightarrow \begin{cases} R=\sqrt{2} \\ \text{and} \\ \theta=\frac{\pi}{4} \text{ OR } \frac{3}{4}\pi \text{ OR } \frac{5}{4}\pi \text{ OR } \frac{7}{4}\pi \end{cases}$$

So the 4 fourth roots of -4 are $z_1=1+i$, $z_2=-1+i$, $z_3=-1-i$ and $z_4=1-i$

Ex. 6 The general solution of the complementary equation is $y_c(x)=C_1e^{-2x}+C_2e^{-x}$ where $C_1,C_2\in\mathbb{R}$

For the particular solution we make the assumption $y_p(x)=(a+bx)e^x$, then $y_p'(x)=be^x+(a+bx)e^x$

and $y_p''(x)=2be^x+(a+bx)e^x$

If we plug these terms into the DE we obtain $5be^x+6(a+bx)e^x=6xe^x$

$\Rightarrow 5b+6a=0$ and $6b=6$

$\Rightarrow b=1$ and $a=-5/6$

As a consequence $y_p(x)=\left(-\frac{5}{6}+x\right)e^x$

and the general solution of the given DE is

$y(x)=\left(-\frac{5}{6}+x\right)e^x+C_1e^{-2x}+C_2e^{-x}$ where $C_1,C_2\in\mathbb{R}$