Uitwerking

Ex. 1: Put the LDE in standard form first. Then we have:

$$y' + \frac{1}{x+1} y = 5x^2$$

An integrating factor is $e^{\int \frac{1}{x+1} dx} = e^{\ln(x+1)} = x+1$, so solve:

$$(x+1)y' + y = 5x^2(x+1)$$

$$\Leftrightarrow \quad (y \cdot (x+1))' = 5x^3 + 5x^2$$

$$\Leftrightarrow \quad y \cdot (x+1) = \frac{5}{4}x^4 + \frac{5}{3}x^3 + C \quad \text{where } C \in \mathbb{R}$$

$$\Leftrightarrow y(x) = \frac{5}{x+1}\left(\frac{1}{4}x^4 + \frac{1}{3}x^3\right) + \frac{C}{x+1} \quad \text{where } C \in \mathbb{R}$$

Ex. 2: This DE is separable, so separate the variables:

$$y \frac{dy}{dx} = \frac{x}{1+x^2}$$

Integrating on both sides yields:

$$\int y \frac{dy}{dx} dx = \int \frac{x}{1+x^2} dx$$

$$\Rightarrow \frac{1}{2}y^2 = \frac{1}{2}\ln(1+x^2) + C_1 \quad \text{where } C_1 \in \mathbb{R}$$

$$\Rightarrow \quad y(x) = \pm\sqrt{\ln(1+x^2) + C} \quad \text{where } C \in \mathbb{R}$$

Imposing the IC $y(0) = -2$ we obtain $y(x) = -\sqrt{\ln(1+x^2) + 4}$

Ex. 3: $z^3 + i = 0 \iff z^3 = -i$

Write both z and $-i$ in polar form,

so $z = R(\cos\theta + i\sin\theta)$ and

$-i = \cos(\frac{3}{2}\pi) + i\sin(\frac{3}{2}\pi)$

Substituting this and using de Moivre's theorem we get:

$$R^3(\cos(3\theta) + i\sin(3\theta)) = \cos(\tfrac{3}{2}\pi + i\sin(\tfrac{3}{2}\pi))$$

$$\iff \begin{cases} R^3 = 1 \implies R = 1 \\ 3\theta = \frac{3}{2}\pi + k2\pi \quad (k \in \mathbb{Z}) \end{cases}$$

$$\implies \theta = \tfrac{1}{2}\pi + k\tfrac{2}{3}\pi \quad (k \in \mathbb{Z})$$

$$\implies \theta = \tfrac{1}{2}\pi \text{ OR } \theta = \tfrac{7}{6}\pi \text{ OR } \theta = \tfrac{11}{6}\pi$$

So $z^3 + i = 0$ has the 3 solutions:

$$z_1 = \cos(\tfrac{1}{2}\pi) + i\sin(\tfrac{1}{2}\pi) = i$$

$$z_2 = \cos(\tfrac{7}{6}\pi) + i\sin(\tfrac{7}{6}\pi) = -\tfrac{1}{2}\sqrt{3} - \tfrac{1}{2}i$$

$$z_3 = \cos(\tfrac{11}{6}\pi) + i\sin(\tfrac{11}{6}\pi) = \tfrac{1}{2}\sqrt{3} - \tfrac{1}{2}i$$

Ex. 4: Substituting $y(x) = e^{Rx}$ we get the characteristic equation $R^2 + 2aR + (a^2+1) = 0$

$\iff (R+a)^2 = -1 \iff R_{1,2} = -a \pm i$

So the general solution of the DE is:

$$y(x) = C_1 e^{(-a+i)x} + C_2 e^{(-a-i)x}$$

$$= D_1 e^{-ax}\cos(x) + D_2 e^{-ax}\sin(x)$$

If we restrict ourselves to real solutions the constants D_1 and D_2 are real.

Ex. 5 : a) $\underline{b}$ is in the range of T

$\Leftrightarrow$ $T(\underline{x}) = \underline{b}$ is consistent.

So consider:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & -1 & -2 \\ 3 & 6 & \beta & 5 \end{array}\right] \; -3 \; \sim$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & \beta-3 & 2 \end{array}\right] \; -1 \; \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & -4 & -4 \\ 0 & 0 & \beta-3 & 2 \end{array}\right] \times\left(-\frac{1}{4}\right)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & \beta-3 & 2 \end{array}\right] \; -(\beta-3) \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 5-\beta \end{array}\right]$$

So : $\underline{b}$ is in the range of $\Leftrightarrow$ $\beta = 5$

b) The standard matrix for T is

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & -1 \\ 3 & 6 & \beta \end{bmatrix}$$

Now it follows:

T is one-to-one $\Leftrightarrow$

each column of A has a pivot position

$\Leftrightarrow$ $\beta \in \mathbb{R}$ (look at part a))

So for each $\beta \in \mathbb{R}$ transformation T

is one-to-one

Ex. 6: Let $\underline{x} \in H$, then:

$$\underline{x} = \begin{bmatrix} r+3s+4t \\ -s-t \\ -r-t \\ 2r+4s+6t \end{bmatrix} = r\begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix} + s\begin{bmatrix} 3 \\ -1 \\ 0 \\ 4 \end{bmatrix} + t\begin{bmatrix} 4 \\ -1 \\ -1 \\ 6 \end{bmatrix}$$

for certain $r, s, t \in \mathbb{R}$

So $H = \operatorname{Span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}}_{\underline{h}_1}, \underbrace{\begin{bmatrix} 3 \\ -1 \\ 0 \\ 4 \end{bmatrix}}_{\underline{h}_2}, \underbrace{\begin{bmatrix} 4 \\ -1 \\ -1 \\ 6 \end{bmatrix}}_{\underline{h}_3} \right\}$

Since $\underline{h}_3 = \underline{h}_1 + \underline{h}_2$, vector $\underline{h}_3$ is redundant. And as $\{\underline{h}_1, \underline{h}_2\}$ is obviously linearly independent, $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \\ 4 \end{bmatrix} \right\}$ is a basis for H.

Ex. 7: a) False, since

B has 3 pivot columns, so $\operatorname{rank}(B) = 3$.

Using the rank theorem we find $\dim(\operatorname{Nul}(B)) = 2$ ⊠

b) True, since

B is the inverse of A^2, so $A^2 B = I_n$.

This implies $A(AB) = I_n$, and using the invertible matrix theorem we conclude A is invertible.

(Note: A and AB are $n \times n$-matrices)

Multiplying both sides by A^{-1} we obtain:

$(A^{-1}A)(AB) = A^{-1} I_n$, so $AB = A^{-1}$ ⊠

c) True, since

let $\underline{h} \in H$, then $\underline{h} = d_1\underline{a} + d_2\underline{b} + d_3\underline{c}$ for certain weights $d_1, d_2, d_3 \in \mathbb{R}$.

Consider:

$\underline{x} \cdot \underline{h} = \underline{x} \cdot (d_1\underline{a} + d_2\underline{b} + d_3\underline{c})$

$= \underline{x} \cdot (d_1\underline{a}) + \underline{x} \cdot (d_2\underline{b}) + \underline{x} \cdot (d_3\underline{c})$

$= d_1(\underline{x} \cdot \underline{a}) + d_2(\underline{x} \cdot \underline{b}) + d_3(\underline{x} \cdot \underline{c})$

$= d_1 \cdot 0 + d_2 \cdot 0 + d_3 \cdot 0 = 0$

So $\underline{x} \perp \underline{h}$ and as a result $\underline{x} \in H^{\perp}$ ⊠

Ex. 8: C is a 3×6-matrix, so the columns of C are vectors in $\mathbb{R}^3$ and as a consequence $\mathrm{COL}(C)$ is a subspace of $\mathbb{R}^3$.

Using the rank theorem and the information that $\dim(\mathrm{NUL}(C)) = 4$ it follows that $\mathrm{rank}(C) = \dim(\mathrm{COL}(C)) = 2$

So $\mathrm{COL}(C)$ is a 2-dimensional subspace of $\mathbb{R}^3$ ($p = 2$ and $q = 3$)

Ex. 9: a) Construct an orthogonal basis $\{\underline{c}_1, \underline{c}_2\}$ for W first.

$\underline{c}_1 = \underline{b}_1$

and $\underline{c}_2 = \underline{b}_2 - \left(\frac{\underline{b}_2 \cdot \underline{c}_1}{\underline{c}_1 \cdot \underline{c}_1}\right) \underline{c}_1 = \underline{b}_2 - \frac{9}{9} \underline{c}_1$

$$= \begin{bmatrix} 1 \\ 0 \\ 2 \\ -2 \end{bmatrix}$$

So $\text{proj}_W \underline{y} = \left(\frac{\underline{y} \cdot \underline{c}_1}{\underline{c}_1 \cdot \underline{c}_1}\right) \underline{c}_1 + \left(\frac{\underline{y} \cdot \underline{c}_2}{\underline{c}_2 \cdot \underline{c}_2}\right) \underline{c}_2$

$$= \frac{18}{9} \underline{c}_1 + \frac{9}{9} \underline{c}_2 = \begin{bmatrix} 5 \\ 4 \\ 2 \\ 0 \end{bmatrix}$$

b) $\underline{x} \in W^\perp \Leftrightarrow \underline{x} \perp \underline{b}_1$ and $\underline{x} \perp \underline{b}_2$

$$\Leftrightarrow \begin{cases} 2x_1 + 2x_2 + x_4 = 0 \\ 3x_1 + 2x_2 + 2x_3 - x_4 = 0 \end{cases}$$

$$\Leftrightarrow \begin{bmatrix} 2 & 2 & 0 & 1 \\ 3 & 2 & 2 & -1 \end{bmatrix} \underline{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \underline{x} \in \text{Nul}\left(\begin{bmatrix} 2 & 2 & 0 & 1 \\ 3 & 2 & 2 & -1 \end{bmatrix}\right)$$

So define $F = \begin{bmatrix} 2 & 2 & 0 & 1 \\ 3 & 2 & 2 & -1 \end{bmatrix}$, then

$W^\perp = \text{Nul}(F)$

Note: You can also take $F = \begin{bmatrix} 2 & 2 & 0 & 1 \\ 1 & 0 & 2 & -2 \end{bmatrix}$, in fact there are infinitely many possibilities.

c) $W^{\perp} = \mathrm{NUL}(F)$, so consider:

$$\left[\begin{array}{cccc|c} 2 & 2 & 0 & 1 & 0 \\ 3 & 2 & 2 & -1 & 0 \end{array}\right] \; -1\tfrac{1}{2} \sim$$

$$\left[\begin{array}{cccc|c} 2 & 2 & 0 & 1 & 0 \\ 0 & -1 & 2 & -2\tfrac{1}{2} & 0 \end{array}\right] \; +2 \sim$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 4 & -4 & 0 \\ 0 & -1 & 2 & -2\tfrac{1}{2} & 0 \end{array}\right]$$

$$\Leftrightarrow \begin{cases} x_1 = -2x_3 + 2x_4 \\ x_2 = 2x_3 - 2\tfrac{1}{2}x_4 \\ x_3, x_4 \text{ are free} \end{cases}$$

$$\Leftrightarrow \underline{x} = x_3 \begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -2\tfrac{1}{2} \\ 0 \\ 1 \end{bmatrix} \text{ where } x_3, x_4 \in \mathbb{R}$$

So $W^{\perp} = \mathrm{Span}\left\{ \begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2\tfrac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\}$, and

since $\left\{ \begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2\tfrac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\}$ is linearly independent

this is a basis for $W^{\perp}$

Ex. 10: $\text{proj}_{\underline{b}}\, \underline{c} = \alpha \underline{b} = \left(\frac{\underline{c}\cdot\underline{b}}{\underline{b}\cdot\underline{b}}\right)\underline{b} = \left(\frac{-10}{25}\right)\underline{b}$

use $u^T u$

$= -\frac{2}{5}\underline{b}$

So $\alpha = -\frac{2}{5}$

Ex. 11: Substituting the data points we get:

$$\begin{cases} \alpha = 2 \\ \alpha + \beta + \gamma = 1 \\ \alpha + 2\beta + 4\gamma = 0 \\ \alpha + 3\beta + 9\gamma = 1 \end{cases} \Longleftrightarrow$$

$$A\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix} = \underline{b} \text{ with } A = \begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 1\\ 1 & 2 & 4\\ 1 & 3 & 9\end{bmatrix} \text{ and } \underline{b} = \begin{bmatrix}2\\ 1\\ 0\\ 1\end{bmatrix}$$

The corresponding system of normal equations is:

$$A^TA\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix} = A^T\underline{b} \text{ with } A^TA = \begin{bmatrix}4 & 6 & 14\\ 6 & 14 & 36\\ 14 & 36 & 98\end{bmatrix}$$

and $A^T\underline{b} = \begin{bmatrix}4\\ 4\\ 10\end{bmatrix}$.

For your information: The least-squares solution is: $y = \frac{21}{10} - \frac{19}{10}x + \frac{1}{2}x^2$