Mathematics 1 AESB1110: Exam 2

October 17, 2014

Answers:

1. *Question:* Use the linear approximation to estimate $\sqrt[4]{9999}$ total: 1 p.

   Solution:

$$f(x) = \sqrt[4]{x} = x^{1/4}$$
$$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$

1/2 p.

$$a = 10000, \quad x = 9999$$
$$f(a) = 10000^{1/4} = 10, \quad f'(x) = \frac{1}{4}x^{-3/4}, \quad f'(a) = \frac{1}{4}(10^4)^{-3/4} = \frac{1}{4000}$$
$$\sqrt[4]{9999} \approx L(9999) = 10 + \frac{1}{4000}(9999 - 10000) = 10 - \frac{1}{4000}$$

1/2 p.

2. *Question:* Write down the first four nonzero terms of the Maclaurin series for the following functions: total: 2 p.

$$f(x) = x^2 e^{-x^2}$$
$$g(x) = \frac{x}{\sqrt{4 + x^2}}$$

   Solution: The Maclaurin series is defined as

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

1 p.

Using the standard formula list we find the first four nonzero terms as

$$\begin{aligned} e^x &\approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \\ e^{-x^2} &\approx 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} \\ f(x) = x^2 e^{-x^2} &\approx x^2\left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!}\right) = x^2 - x^4 + \frac{1}{2}x^6 - \frac{1}{6}x^8 \end{aligned}$$

1/2 p.

$$\begin{aligned} (1+x)^{-1/2} &\approx 1 + \left(-\frac{1}{2}\right)x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}x^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}x^3 \\ &= 1 - \frac{1}{2}x + \frac{3}{(4)2!}x^2 - \frac{3(5)}{(8)3!}x^3 = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 \\ (4+x^2)^{-1/2} &= 4^{-1/2}\left(1 + \frac{x^2}{4}\right)^{-1/2} \approx \frac{1}{2}\left(1 - \frac{1}{2}\left(\frac{x^2}{4}\right) + \frac{3}{8}\left(\frac{x^2}{4}\right)^2 - \frac{5}{16}\left(\frac{x^2}{4}\right)^3\right) \\ &= \frac{1}{2} - \frac{1}{16}x^2 + \frac{3}{256}x^4 - \frac{5}{2048}x^6 \\ g(x) &= \frac{x}{\sqrt{4+x^2}} \approx x\left(\frac{1}{2} - \frac{1}{16}x^2 + \frac{3}{256}x^4 - \frac{5}{2048}x^6\right) \\ &= \frac{1}{2}x - \frac{1}{16}x^3 + \frac{3}{256}x^5 - \frac{5}{2048}x^7 \end{aligned}$$

1/2 p.

Computing the Maclaurin expansion directly (using the derivatives) and not completely finishing the work is worth 1/4 points each. It is also possible to compute the second series via integration. Leaving products of numbers instead of explicit fractions in the end result, e.g, $\frac{1}{2(2)(4)}$ etc, is OK.

3. *Question:* Compute the derivative $f'(x)$ of the function: total 1 p.

$$f(x) = \int_2^{2x+x^2} \tan(t)\,dt$$

Answer: Using the Fundamental Theorem of Calculus and the chain rule we get:

$$f'(x) = \tan(2x+x^2)(2x+x^2)' = (2+2x)\tan(2x+x^2)$$

4. *Question:* Compute the following integrals: 6 p.

$$\int \frac{\sin(\ln x)}{x}\,dx$$

$$\int x^2\sqrt{2+x}\,dx$$

$$\int_0^1 x^3 e^{-x^2}\,dx \quad \text{(For this one you need both substitution and integration by parts)}$$

Answer:

$$\int \frac{\sin(\ln x)}{x}\,dx = \begin{bmatrix} u = \ln x \\ du = \frac{1}{x}\,dx \end{bmatrix} = \int \sin u\,du = -\cos u + C$$
$$= -\cos(\ln x) + C$$

2 p.

$$\int x^2\sqrt{2+x}\,dx = \begin{bmatrix} u = 2+x \\ du = dx \\ x^2 = (u-2)^2 \end{bmatrix} = \int (u-2)^2\sqrt{u}\,du$$
$$= \int (u^2 - 4u + 4)u^{1/2}\,du = \int (u^{5/2} - 4u^{3/2} + 4u^{1/2})\,du$$
$$= \frac{1}{5/2+1}u^{5/2+1} - \frac{4}{3/2+1}u^{3/2+1} + \frac{4}{1/2+1}u^{1/2+1} + C$$
$$= \frac{2}{7}u^{7/2} - \frac{8}{5}u^{5/2} + \frac{8}{3}u^{3/2} + C$$
$$= \frac{2}{7}(2+x)^{7/2} - \frac{8}{5}(2+x)^{5/2} + \frac{8}{3}(2+x)^{3/2} + C$$

2 p.

$$\int_0^1 x^3 e^{-x^2}\,dx = \begin{bmatrix} u = x^2 \\ du = 2x\,dx \\ x^2 = u \\ u(0) = 0, \quad u(1) = 1 \end{bmatrix} = \frac{1}{2}\int_0^1 ue^{-u}\,du$$
$$= \begin{bmatrix} f = u & f' = 1 \\ g' = e^{-u} & g = -e^{-u} \end{bmatrix} = \frac{1}{2}\left(\left[-ue^{-u}\right]_0^1 - \int_0^1 \left(-e^{-u}\right)\,du\right)$$
$$= \frac{1}{2}\left((-1)e^{-1} - (-0)e^{-0} - \left[e^{-u}\right]_0^1\right) = \frac{1}{2}\left(-\frac{1}{e} - \left(\frac{1}{e} - 1\right)\right)$$
$$= \frac{1}{2}\left(1 - \frac{2}{e}\right) = \frac{1}{2} - \frac{1}{e}$$

2 p.