XIII. Analysis of complex transport problems

A. Motivation

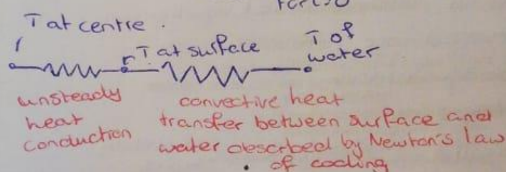
Although you learn (I hope!) a number of specific analytical and mathematical tools for solving transport problems in this course, most problems in the real world don't fit any of the tools you have learned - or will ever learn. Consider two simple examples:

1. Cooling of hot metal ball in cold water

"modes in series"

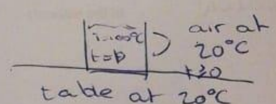
water at
t=0°C
For t>0

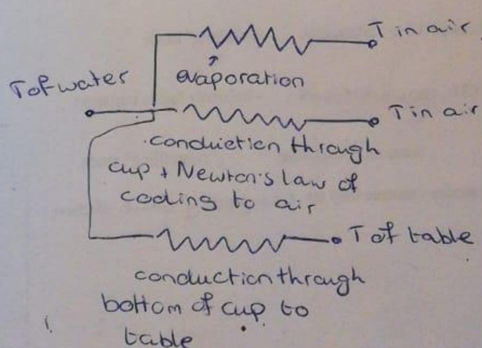
what is T of ball
as function of time?



2. Cooling of cup of coffee

"modes in parallel"





B. method:

1. Diagram overall transport process in terms of its individual "modes"

   - modes "in series"
   - modes "in parallel"

2. Analyze the problem

   - for modes in series, assuming any mode is at equilibrium always gives an overestimate of the rate of heat transfer
     - i.e., faster equilibration in the estimate than in the real system
   - for modes in parallel, assuming any mode is completely shut down always gives an underestimate of the rate of heat transfer
     - i.e., slower equilibration in the estimate than in the real system

3. Simplify the problem:

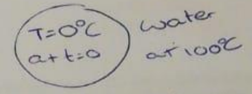
   For modes in series, focus on the slowest, "controlling" mode

   - assume all other modes are at equilibrium - infinitely fast in transport

   For modes in parallel, focus on fastest, "controlling" mode

   - ignore transport from slower modes - assume they are completely shut down

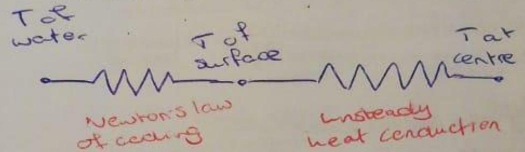
4. Solve simplified transport problem



D. worked examples

1. **Iron ball,** Dia. = 0.1 m. initial T = 0°C at t = 0. At t ≥ 0, ball is dropped into boiling water with T maintained at 100°C. h = 500 W/(m² K) at surface of ball. How long until T reaches 90°C at center of ball? For iron, k = 73 W/(m K), ρ = 7880 kg/m³, and $\hat{C}_p$ = 511 J/(kg K).

method: **steps (1) and (2): Diagram and analyze problem:**



modes:

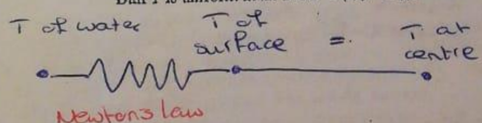
1) Newton's law of cooling at surface

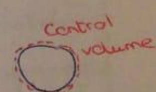
2) unsteady conduction ~~is~~ inside

- modes "in series"

step 3) Simplify problem.

Approximate solution (#1). Assume mode 1 "controls." For modes in series, this means that mode 2 (internal conduction) is at equilibrium. Ball T is uniform at all times. T(r,t) = T(t) in ball.





step 4) Solve simplified problem. Perform macroscopic energy balance

system: ball

$$\underbrace{\underbrace{4\pi R^2}_{\text{area}} h(T_w - T)}_{\text{energy in}} = \underbrace{\underbrace{\tfrac{4}{3}\pi R^3}_{\text{volume}} \rho \hat{C}_p \left(\frac{dT}{dt}\right)}_{\text{accumulation}}$$

$$= \tfrac{4}{3}\pi R^3 \rho \hat{C}_p \frac{d}{dt}(T_w - T)$$

$$\frac{d(T_w - T)}{T_w - T} = \frac{4\pi R^2 h}{4/3\,\pi R^3 \rho c_p}\, dt$$

$$\ln(T_w - T) = \frac{4\pi R^2 h}{4/3\,\pi R^3 \rho c_p} \cdot t + c_1 \qquad T_w - T = 100 - 0 \text{ at } t = 0$$

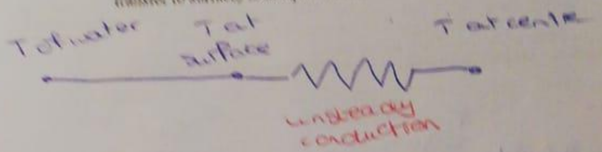
$$\frac{T_w - T}{T_w - T_0} = \exp\left[\left(-\frac{3ht}{R\rho\hat{C}_p}\right)\right] = \exp(-0{,}0074t)$$

T at centre of ball is 90°C

at 309 s

13.6

back to step (3) Approximate solution (#2). Assume mode (2) "controls." For modes in series, that means that mode (1) (convective heat transfer to surface) is at equilibrium: T of water = T at surface of ball.



step 4) Solve simplified problem.

Back Fig 11.5-3

T at $\frac{r}{R}=0$ is 90°C at 35 s

step 5) check solutions.

Approximate solution (#1) predicts slower heat transfer. For modes in series, *any* approximate solution that leaves out one mode gives *faster* equilibration than the real process. Therefore, approximate solution (#1), which evidently errs less in this direction, is the better answer.

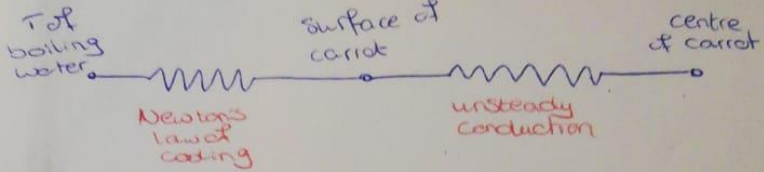
real time > approx solution 1 >> approx solution 2

step 6) estimate effect of simplifying problem.

For modes in series, leaving out *any* mode can only err towards giving faster equilibration than is correct. Therefore, the true answer is *slower* equilibration, or a longer time to come to 90°C, than estimated above.

However, since approximate solution (#1) predicts *much* slower equilibration than the other solution, mode 2 is evidently relatively unimportant in this overall process. Therefore approximate solution (#1), which errs only in leaving out insignificant mode 2, is probably nearly correct.

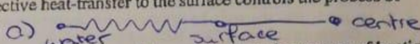
real time > 309 s



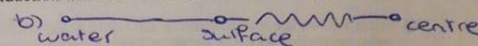
13.

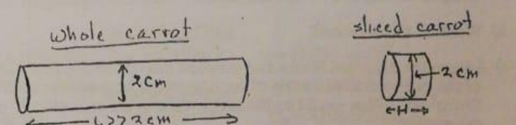
Rocky is cooking carrots in boiling water. The carrots are initially at 0°C (the refrigerator temperature) and must be heated to 80°C at the center, in water that is maintained at 100°C. Rocky usually prepares the carrots by boiling them whole, and they then take 10 minutes to reach the desired temperature. A whole carrot is a long cylinder, about 2 cm in diameter. Rocky wants to slice the carrots into cylindrical disk shapes, with diameter still 2 cm, but with thickness H, so that the carrot pieces will reach the desired temperature in only 5 minutes. What value of H should he pick if

a) he assumes that convective heat-transfer to the surface controls the process of heating of the carrot?



b) he assumes that conduction within the solid carrot controls the process of heating of the carrot?





a) An energy balance for both cases is $V\rho\hat{C}_p \frac{dT}{dt} = -h(T-T_w)A$
$\Rightarrow dT/dt = -[hA/(V\rho\hat{C}_p)](T-T_w) \rightarrow \ln(T-T_w) = -[hAt/(V\rho\hat{C}_p)] + C$, or
$(T-T_w) = C' \exp\{-[hAt/(V\rho\hat{C}_p)]\}$. Since $T-T_w$ is initially 100K, $C' = 100$.
If one wants the same $(T-T_w)$ in half the time, one must compensate so the factor $hAt/(V\rho\hat{C}_p)$ is unchanged. h and $\rho\hat{C}_p$ are fixed, so the change has to come in the ratio of A to V.
For the whole carrot, $A = 2\pi R^2 + 2\pi RL \cong 2\pi RL$ if $L >> R$; $V = \pi R^2 L$, so
$hAt/V\rho\hat{C}_p = (h/\rho\hat{C}_p)[2\pi RLt/(\pi R^2 L)] = (h/\rho\hat{C}_p)(2t/R)$.
For the sliced carrot, $A = 2\pi R^2 + 2\pi RH$, $V = \pi R^2 H$, and $t' = \frac{1}{2}t$.
$hAt/(V\rho\hat{C}_p) = (h/\rho\hat{C}_p)[(2\pi R^2 + 2\pi RH)\frac{1}{2}t/(\pi R^2 H)]$. The factor in brackets should equal $2t/R$ as for the whole carrot:

$$\frac{(2\pi R^2 + 2\pi RH)t}{2\pi R^2 H} = \frac{2t}{R} \rightarrow \frac{(R/H+1)t}{R} = \frac{2t}{R} \rightarrow R/H = 1 \text{ or } H = R = 0.01 \text{ (1 cm)}$$

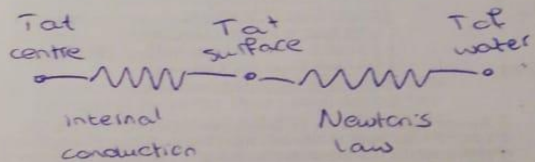
b) For the whole carrot, since $L >> R$, this is an "infinite" cylinder, and conduction reflects Fig 12.1-2. Since we want $T = 80°C$, with $T_1 = 100°C$ and $T_0 = 0°C$, we want either $(T-T_0)/(T_1-T_0) = 0.8$ or $(T_1-T)/(T_1-T_0) = 0.2$ at $r = 0$.
From Fig 12.1-2, we need $\alpha t/R^2 \cong 0.36$.
Now with the sliced carrot, the solution will be a product of slab + cylinder. For the cylinder, α and R are unchanged, and t decreases by $\frac{1}{2}$. Therefore $\alpha t/R^2 \cong 0.18$. From Fig 12.1-2, $(T_1-T)/(T_1-T_0) \cong 0.54$. (For this case, since product method is involved, one <u>must</u> use $(T_1-T)/(T_1-T_0)$.) We still need $\frac{T_1-T}{T_1-T_0} = 0.2$ for carrot slice. Therefore $(0.2) = (0.54)\left(\frac{T_1-T}{T_1-T_0}\right)_{slab} \Rightarrow \left(\frac{T_1-T}{T_1-T_0}\right)_{slab} = 0$
From Fig 12.1-1, $(T_1-T)/(T_1-T_0) = 0.37$ at $b = 0$, if $\alpha t/b^2 \cong 0.5$
Now if $\frac{\alpha t}{b^2} = 0.5$ and $\frac{\alpha t}{R^2} = 0.18$, then $0.5b^2 = 0.18R^2 \Rightarrow b = (0.18/0.5)^{1/2}$
$b = 0.6R$; $2b = H = (1.2)(0.01\text{ m}) = 0.012\text{ m} = 1.2\text{ cm}$

~~could~~ could also use BSLK Fig. 11.5-2

E. a final note

Determining which mode is fastest or slowest can be ambiguous.

For instance, consider a solid sphere dropped in water. There are two modes in series.



At t → 0, by itself, the internal conduction formula (Fig. 12.1-3) predicts an infinite rate of heat transfer

BSLK eq. 11.5-12

(recall eq. 12.1-10: $q_y|_{y=0} = \frac{k}{\sqrt{\pi \alpha t}} (T_1 - T_0)$)

∴ Newton's law of cooling at the surface *always* controls in the first instant of contact. At these short times, Newton's law isn't able to keep up with the huge capacity of internal conduction with a sharp temperature gradient at the surface. Therefore at short time, Newton's law is the "bottleneck,", and controls the heat-transfer process.

If convective heat transfer to the surface is relatively efficient, then control soon shifts to internal conduction mode; at long times, Fig. 12.1-3 is reasonably accurate.

11.5-3 BSLK

Lesson: approach outlined here for "complex" problems is heuristic, not rigorous.