XII. Multivariate and Unsteady Conduction

A. governing partial differential equation and assumptions

- see example 9 of XI.C $\frac{\partial T}{\partial t} = \alpha \nabla^2 T \; ; \; \alpha = \frac{k}{\rho \hat{C}_p}$

Assumptions:

- no heat generation within system (solid)
- no convection within solid
- ~~prop~~ uniform constant $k, \rho, \hat{C}_p$

same mathematical equation is called "diffusivity equation" ~~in PGE 323~~, because

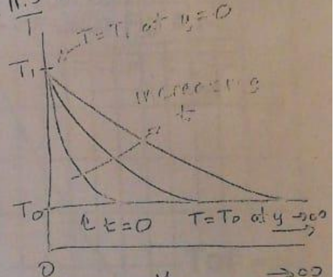
- heat conduction
- chemical diffusion
- Darcy flow in porous media

all obey the same partial differential equation

B. Tabulated 1D solutions
(7 solutions) BSLK Sect 3.8, Ex. 11.5-1

1. Semi-infinite slab (BSL Sect. 4.1, Ex. ~~11~~ 12.1-1)

a. initial and boundary conditions



$T = T_0$ at $t = 0$, for $y \geq 0$

$T = T_1$ at $y = 0$ for $t > 0$

$T = T_0$ at $y \to \infty$ for $t > 0$

b. solution given by Eq. ~~11~~ 12.1-8 and Fig. 4.1-2 BSL K Fig 3.8-2
BSLK Eq. 11.5-10

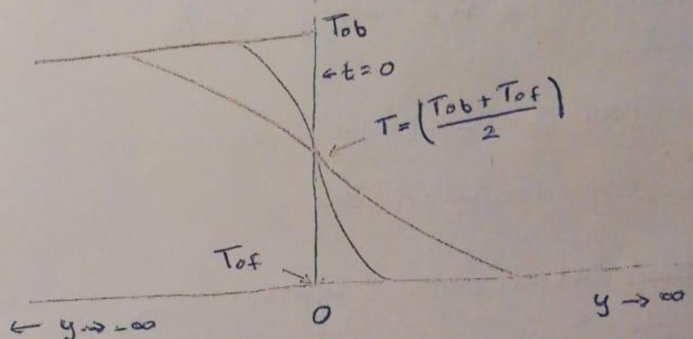
$$\frac{T - T_0}{T_1 - T_0} = 1 - \mathrm{erf}\left[\frac{y}{\sqrt{4\alpha t}}\right]$$

c. heat flux at y = 0

$$q_y\Big|_{y=0} = -k \frac{dT}{dy}\Big|_{y=0} = \frac{k}{\sqrt{\pi \alpha t}}(T_1 - T_0)$$

BSLK 11.5-12

2. **Two semi-infinite slabs brought together**
   - slabs have identical properties, different initial temperatures



a. **boundary and initial conditions**

$T = T_{of}$	at $t = 0$,	for $y > 0$
$T = T_{of}$	at $y \rightarrow \infty$,	for $t > 0$
$T = T_{ob}$	at $t = 0$,	for $y < 0$
$T = T_{ob}$	at $y \rightarrow -\infty$,	for $t > 0$
$T(y \rightarrow 0^+) = T(y \rightarrow 0^-)$		for $t > 0$
$q_y(y \rightarrow 0^+) = q_y(y \rightarrow 0^-)$		for $t > 0$

b. **solution**

$$\left[\frac{T - T_{of}}{T_{ob} - T_{of}}\right] = \frac{1}{2}\left[1 - \operatorname{erf}\frac{y}{\sqrt{4\alpha t}}\right]$$

(note erf(-x) = - erf(x); thus equation applies to y > 0 and
solution is plotted on next page

BSLK Ex 11.5-2

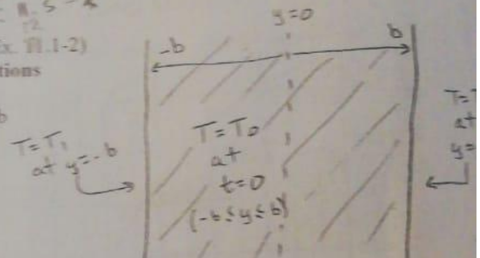
3. **Slab of finite thickness** (BSL 2 Ex. 12.1-2)

a. **boundary and initial conditions**

$T = T_0$ at $t = 0$, for $-b \le y \le b$

$T = T_1$ at $y = b$ for $t > 0$

$T = T_1$ at $y = -b$ for $t > 0$



b. **solution:** BSL Figure. 12.1-1
BSLK Fig 11.5-1

(note curves for small $\alpha t/B^2$ do *not* go below horizontal axis)

c. **an aside: how to estimate heat flux at surface from this chart?**

- estimate $\frac{\partial T}{\partial y}$ from slope at $y/b = 1$

Use Fouriers law to get q_y

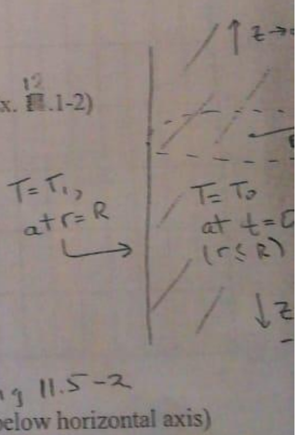
4. **Conduction within cylinder** ($0 \le r \le R$) (BSL 2 Ex. 12.1-2)

a. **boundary and initial conditions**

$T = T_0$ at $t = 0$, for $0 \le r \le R$

$T = T_1$ at $r = R$ for $t > 0$

T finite at $r = 0$ for $t > 0$



b. **solution:** BSL Figure. 12.1-2 BSLK Fig 11.5-2

(note curves for small $\alpha t/R^2$ do *not* go below horizontal axis) and the boundary condition is

$$\frac{d\bar{v}}{dr} = h\bar{v}, \quad \text{at } r = a. \tag{12}$$

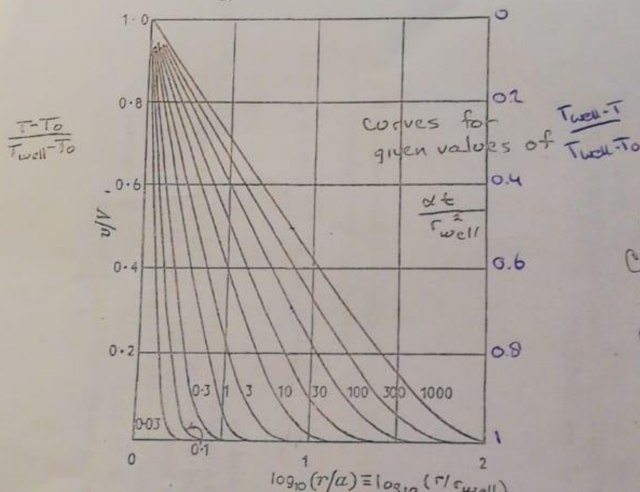


FIG. 41. Temperatures in the region bounded internally by the cylinder $r = a$, with zero initial temperature and constant surface temperature V. The numbers on the curves are the values of $\kappa t/a^2$.

from Carslaw + Jaeger, Conduction of Heat in Solids, 1949

Thus
$$\bar{v} = \frac{V}{p}\left\{1 + \frac{hK_0(qr)}{qK_0'(qa) - hK_0(qa)}\right\}, \tag{13}$$

and
$$v = V + \frac{hV}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\frac{K_0(\mu r)\,d\lambda}{\lambda[\mu K_0'(\mu a) - hK_0(\mu a)]}. \tag{14}$$

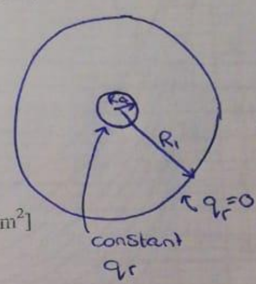
The integrand has a branch point at $\lambda = 0$, so we use the contour of Fig. 40. There are no poles within† or on this contour. Then proceeding as in I above we obtain

$$= -\frac{2hV}{\pi}\int_0^\infty e^{-\kappa u^2 t}\frac{J_0(ur)[uY_1(ua)+hY_0(ua)] - Y_0(ur)[uJ_1(ua)+hJ_0(ua)]}{[uJ_1(ua)+hJ_0(ua)]^2 + [uY_1(ua)+hY_0(ua)]^2}\frac{du}{u}. \tag{15}$$

† Carslaw and Jaeger, loc. cit., or Erdélyi and Kermack, *Proc. Camb. Phil. Soc.* [4]1 (1945) 74.

8. Unsteady radial conduction from a constant heat source in a finite radial solid

In this case, the boundary condition is a constant heat flux, rather than a constant temperature.



Assumptions:

- Constant heat flux q_0 from surface at $r = R_0$;

$$q_0 = \frac{Q}{2\pi R_0 H} \qquad (1)$$

  Q = total heat output of heat source
  q_0 = heat flux at inner surface of solid [W/m^2]
  R_0 = radius heating element [m]
  H = height heating element [m]

- Perfectly insulated outer boundary at $r = R_1$
- Assume no heat flux (perfect insulation) at the top and bottom flat surfaces.
- Uniform and constant properties k, ρ, C_p within the solid

Initial and boundary conditions:

$T = T_0$ at t=0 for all R

$-k\dfrac{\partial T}{\partial r} = q_0$ at $r = R_0$ for t>0 [constant heat flux from heater at R_0]

$-k\dfrac{\partial T}{\partial r} = 0$ at $r = R_1$ for t>0 [perfectly insulated outer boundary at R_1]

k ≡ Thermal conductivity [W m^{-1}K^{-1}]

One needs a separate solution and chart for each ratio (R_1/R_0).

<u>Dimensionless variables</u>

Let $r_D \equiv \dfrac{r}{R_0}$; $1 \le r_D \le \dfrac{R_1}{R_0}$ (2

Let $t_D \equiv \dfrac{\alpha t}{R_0^{\,2}}$ (3

Let $T_D = \dfrac{(T - T_0)k}{R_0 q_0}$ (

On the following pages are plots for two cases, with (R_1/R_0) = 14 and (R_1/R_0) = 10
(Note that this case differs from case 6, which also involves radial conduction, in t
ways: First, it involves a constant heat flux at the inner radius, not a constant
temperature. Second, there is a finite outer boundary, rather than an infinite radial